1 Game Theory

1.1 Normal Form Game

Generally, different players have conflicting interests, i.e. they want to maximize different objective functions. We use game theory to analyze such situations and make predictions given that each player seeks to maximize their utility given others’ actions.

A normal-form simultaneous-move game is defined by:

1. $N = \{1, \ldots, n\}$ agents indexed by $i$

2. $A = A_1 \times \ldots \times A_n$, where $A_i$ is a set of actions available to agent $i$ and $a = (a_1, \ldots, a_n) \in A$ denotes an action profile

3. A payoff matrix or utility function $u$ mapping each combination of actions to a utility for each agent

4. A mixed strategy $s_i$ is a distribution on actions $A_i$

1.2 Solution Concept

A solution concept is a formal rule for predicting how a game will be played. Here are some basic solution concepts:

1. **Best response.** A strategy $s_i$ is a best response to the strategies of other players if agent $i$ maximizes its expected utility.

2. **Dominated Strategy:** Action $a_i$ strictly dominates action $a_i'$ if $u_i(a_i, a_{-i}) > u_i(a_i', a_{-i})$ for all actions $a_{-i}$ of others.

3. **Iterated Elimination of Dominated Strategies:** No player will ever choose a strictly dominated strategy, so we can iteratively eliminate strictly dominated strategies before solving for a Nash equilibrium.

4. **Nash Eqilibrium (NE):** In a $n$-player game, strategy profile $(s_1^*, \ldots, s_n^*)$ is a Nash eqilibrium if $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$ for all players $i$. In other words, every player’s strategy must be a best response to others. Nash equilibria can be pure strategy Nash eqilibria (PSNE) (pure strategies only) or mixed strategy Nash eqilibria (MSNE).
1.3 Sequential-Move Game

In *sequential-move games*, a player’s strategy needs to describe the action she will take after every sequence of observations about the actions of others:

1. **Extensive form of sequential-move game:** The *extensive form* is defined by

   (a) A set of players $N = \{1, \ldots, n\}$
   
   (b) A set $H$ of action histories $h$ (with empty history $\varepsilon$)
   
   (c) A set $Z \subset H$ of terminal histories and a utility $u_i(h)$ for each player $i$ and $h \in Z$
   
   (d) A mapping from each non-terminal history $h \in H \setminus Z$ to a player $i$ and action set $A_i(h)$

2. **Strategies in extensive form:** The strategy $s_i$ in an extensive form game defines $s_i(h) \in A_i(h)$ for every non-terminal history $h \in H \setminus Z$ for which it is agent $i$’s turn to play. (Analogy: a player writes down her contingency plan for every situation before the game starts.)

3. **Subgame-perfect Equilibrium:** A strategy profile $s^* = (s^*_1, \ldots, s^*_n)$ is a *subgame-perfect equilibrium* if it is a Nash equilibrium in the subgame at every non-terminal history $h \in H \setminus Z$.

4. **Single-deviation Principle:** A strategy profile is a subgame-perfect equilibrium in a finite extensive form game *iff* there is no subgame at non-terminal history $h$ where the agent whose turn it is to move has a useful deviation by changing his action at that history $h$ only.

1.4 Repeated Game

In a *repeated game* $G^T$, the same simultaneous move *stage game* $G = (N, A, u)$ is played by the same players for $T$ periods, with every agent having perfect information about the history of actions. In an infinitely repeated game $G^\infty$ the stage game $G$ is repeated forever:

1. **Utility and discounting:** A player’s total utility is the sum of his utilities across stage games. In this sum, a player’s utility from game $k$ is discounted $\delta^k$, for $\delta \in [0, 1)$.

2. **Single-deviation Principle for Repeated Games:** A strategy profile is a subgame-perfect equilibrium in a (infinitely-) repeated game *iff* for every subgame, no agent can improve her utility by changing her action in the current period and leaving the rest of the strategy unchanged.

3. **Unique subgame-perfect equilibrium:** If the stage game has a unique Nash equilibrium, then a finitely repeated game has a unique subgame-perfect equilibrium, which is to play the stage game Nash equilibrium in every period.

4. **Open-loop strategy:** An *open-loop strategy* $s_i$ for player $i$ in a repeated game satisfies $s_i(h) = s_i(h')$ for all histories $h, h'$ of the same length. An open-loop strategy profile in which a stage game Nash equilibrium is played in each period is a subgame-perfect equilibrium of the repeated game.

5. **Automaton strategy:** An *automaton strategy* $m_i$ for player $i$ is defined by

   (a) A set $Q_i$ of machine states
(b) A start state \( q_0^i \in Q_i \)
(c) A next state function \( q'_i = \text{succ}_i(q_i, a) \) for all states \( q_i \) and action profiles \( a \in A \)
(d) A mapping from states to actions \( f_i : Q_i \to A_i \).

1.5 Exercises\(^1\)

1. Three politicians are running in an election. Before the election, they participate in a pairwise debate. Every politician debates with each opponent separately, so that there are 3 debates in total. Before the debate series, each politician chooses an amount of negative campaigning (\( a, b \) or \( c \)), which cannot change during the election campaign. The outcomes of any pairwise debate are shown in the matrix below (you can think of them as the number of votes gained by the politicians as a result of a debate).

<table>
<thead>
<tr>
<th></th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>10,10</td>
<td>6,9</td>
<td>3,8</td>
</tr>
<tr>
<td>( b )</td>
<td>9,6</td>
<td>5,5</td>
<td>2,4</td>
</tr>
<tr>
<td>( c )</td>
<td>8,3</td>
<td>4,2</td>
<td>1,1</td>
</tr>
</tbody>
</table>

(a) Find all Nash equilibria of the game where the politicians care only about the total number of votes gained.

(b) Now suppose that politicians’ sole objective is to win the election, i.e. to get the largest total number of votes obtained in two debates. The tie-breaking rule for the election is randomization with equal probabilities. Is there a Nash equilibrium in which everyone plays the same strategy you found in part (a)? Find all Nash equilibria.

1. There are 196 countries in the world. Simultaneously, each country \( i \) chooses a production level \( x_i \), incurring cost \( c_i x_i^2 \) for some \( c_i > 0 \). The country \( i \) receives an increase \( x_i \) in its GDP, but this also pollutes the environment and decreases the well-being of every country by \( \alpha \cdot x_i \) for some \( \alpha \in (0, 1) \). The resulting payoff of player \( i \) is

\[
u_i(x_1, \ldots, x_n) = x_i - \alpha \sum_{j=i}^{n} x_j - c_i x_i^2\]

(a) Is there a dominant strategy equilibrium? If yes, compute it; if no, explain why.
(b) Compute the \( x = (x_1, \ldots, x_n) \) vector that maximizes \( \sum_i u_i(x) \). Compare with the answer to (b).

2 Algorithmic Game Theory

2.1 Review

1. Two player zero sum games
   - Maximin Strategy: \( s_1 \in \arg \max_{s_1} \left[ \min_{a_2 \in A_2} u_1(s_1, a_2) \right] \)

\(^1\)Some exercises taken from Fudenberg’s Economics 1052: Game Theory
• Minimax Strategy: $s_1 \in \arg \min_{s_1} \left[ \max_{a_2 \in A_2} u_2(s_1, a_2) \right]
• Maximin and minimax value
• Minimax Theorem: In any two-player, zero-sum game,
  – For each player the set of maximin strategies is identical to the set of minimax strategies.
  – Any maximin strategy for player 1 and any maximin strategy for player 2 is a NE, and these correspond to all NE’s.
  – Maximin value = minimax value = expected utility.

2. Equilibrium of Two Player General Sum Games: This can be found using linear feasibility programs given a pair of candidate supports. Because one needs to check all possible sets of supports, and the number of possible supports is exponential in the number of actions, this algorithm has an exponential run time.

3. The problem of finding a NE (Nash) in General Sum Games is PPAD-Complete. This complexity class includes problems that always have a solution and where the solution can be found by walking on a potentially exponentially long path in a graph. It is likely that PPAD-hard problems cannot be solved in worst-case polynomial time.

4. Complexity classes:
   • P: yes-no decision problems that can be solved in worst case polynomial time, e.g. linear programming, 2SAT, max flow
   • NP: yes instances can be verified in polynomial time
   • NP-hard: “very hard” problems, not necessarily in NP, and at least as hard as the hardest in NP; e.g. the subset-sum problem
   • NP-complete: problems in NP and NP-hard (any problem in NP can be reduced to it in polynomial-time), e.g. 3SAT
   • PPAD: total search problems that can be reduced to End-of-the-Line.
   • PPAD-hard: problems to which End-of-the-Line can be reduced
   • PPAD-complete: problems in PPAD and PPAD-hard

5. Nash is PPAD-hard.

6. Correlated Equilibrium: Can be computed in worst-case polynomial time in the size of the payoff matrix. Unlike in n-player general sum game, where we are multiplying probabilities (and thus non-linear), the probability of a joint action is encoded as a single variable, so we are able to formulate linear feasibility programs.

2.2 Exercise

1. Explain how gadgets relate to algorithmic game theory.

2. Design a 4-agent gadget, where $x_1, x_2, x_4 \in [0, 1]$ are the mixed strategies of agents 1, 2 and 4, respectively, to implement $x_4 = \max(x_1 - x_2, 0)$. Agent 4 represents the output, with payoff 1 when it takes the opposite action to agent 3, and 0 otherwise. Agents 1 and 2 represent
the inputs, and have zero payoff for all actions of the other agents. Design the payoff for the middle agent 3, and prove (for any fixed $x_1, x_2$) that the required output property holds in every Nash equilibria.

3 Auction Theory

3.1 Review

1. Auctions: The possible goals of the designer:
   - Efficiency: allocate the item to the agent with the highest value
   - Revenue: Maximize expected revenue for the seller (design an optimal auction)

2. Value settings:
   - Private value: agent knows own value, value unaffected by value or information of others.
     Special case is independent Private Values (IPV): independent and identically distributed (iid) valuations (e.g. uniform distribution)
   - Interdependent values: belief about an agent’s value may depend on value and information of others. Common value: item has same value to all, but uncertain (and beliefs about value may depend on value and information of others).

3. Quasi-linear utility and it’s role in auctions. [Value can be quantified in monetary units, thought of as “willingness to pay.”]

4. Bayes-Nash equilibrium (applied to auctions):
   Strategy profile $s^* = (s^*_1, \ldots, s^*_n)$ is a BNE in a sealed bid auction if for all agents $i$ and values $v_i$, 
   \[
   E_{v_{-i}} [u_i(s^*_i(v_i), s^*_{-i}(v_{-i}))] \geq E_{v_{-i}} [u_i(b_i, s^*_{-i}(v_{-i}))], \text{ for all bids } b_i. \tag{1}
   \]

5. Interim allocation, payments: the interim allocation $x^*_i(v_i)$ is the probability of being allocated in equilibrium if value $v_i$; the interim payment $t^*_i(v_i)$ is the expected payment in equilibrium if value $v_i$ (considering distribution on others’ values, allowing for both winning and losing the auction).

6. Second-price sealed-bid auction: The bidder with highest bid wins the item and pays the second-highest bid. Bidding true value is a dominant strategy. The auction is efficient in the truthful equilibrium.

7. First-price sealed-bid auction: The bidder with highest bid wins the item and pays her bid value. The BNE in IPV with uniform $U(0,1)$ values is $s^*(v_i) = \frac{n-1}{n} v_i$, where $n$ is the number of agents. Strategy symmetric and strict increasing, so auction is efficient.

8. Revenue Equivalence Theorem: all auctions with the same interim allocation have the same expected revenue (in equilibrium); e.g., FPSB and SPSB (in truthful equil.) have same expected revenue. This characterization can also be used for computing equilibria of non-standard auctions.
3.2 Exercise
Consider a first-price sealed bid auction, where each bidder i has value \( v_i \), sampled IID from \( U(0, 1) \).

Depart from the quasi-linear model. Assume instead that if bidder i wins with bid \( b_i \), her utility is \( (v_i - b_i)^{1/m} \) where \( m > 1 \).

1. Show there is a symmetric pure strategy Bayesian-Nash equilibrium in which each bidder uses the strategy \( s^*_i(v_i) = \left( 1 - \frac{1}{m(n-1)+1} \right) v_i \).

2. How does the seller’s expected revenue from this auction compare to the symmetric equilibrium of the FPSB auction with the same distribution of values, and with the standard quasi-linear utility of \( v_i - b_i \)?

3. How do the expected revenues from the FPSB and the SPSB auction compare when the utility functions of the bidders are as in part (a)?

4 Peer-to-Peer Systems

4.1 Review
BitTorrent: Torrent files contain metadata for a file, and the IP of a tracker. That tracker holds information about the swarm, the group of users currently downloading/uploading that file. Users can request a list of roughly 50 peers from the tracker, to begin downloading the file.

Peers exchange information with one another about which pieces of the file they have. Each person will download from any peers that will upload to them. But, they will only upload to those peers that meet certain conditions.

In the reference client, there are four upload slots. The first 3 slots are given to the peers from whom a peer has the highest download rate. The 4th is for ‘optimistic unchoking’, and given to a random peer with the hope that the peer will reciprocate.

Exploits: under-reporting pieces available for upload, strategic unchoking and maximize “return on investment”, uploading garbage (effectively prevented by hashing blocks), growing neighborhood quickly and free-riding (effectively prevented by having trackers limit multiple requests from the same IP).

4.2 Exercise

1. Explain why breaking files into small pieces is helpful.

2. How might you measure the performance of a swarm? Of a client protocol?

3. Why is optimistic-unchoking helpful for the system?

4. What does cooperation mean, and is this desirable for the system?
5 Reputation Mechanisms

5.1 Review

5.1.1 Key Points

1. **Reputation systems** aggregate feedback from users’ past experiences, to enable other users to make more informed decisions in the future.

2. **Adverse selection** arises in situations with asymmetric information between the agents involved in a transaction, and agents that vary in intrinsic quality. If severe, it can lead only low quality agents to enter a market (and a market can break down).

3. **Moral hazard** arises in situations where an agent has the opportunity to deviate from a promised course of action and without bearing the full (negative) consequences of the action. If the percentage of cheaters becomes too large, a market might break down completely, because the risk for the buyers might become too large.

4. **The reputation game**: Consider a group \( n \) agents (\( n \) even for simplicity), where in each round agents are paired up randomly to play the Prisoners’ Dilemma game. How do the strategies for two-player PD translate to this setting? How can reputation systems affect the equilibria?

5. **Whitewashing attacks** consist of an agent exiting the system when she has a bad reputation, creating a new identity to start afresh. This can be combated by raising the bar to entry, for instance requiring new users to verify accounts with a photo ID, or through “pay your dues” kinds of responses that seek to make the system less useful for newcomers.

6. **eBay’s old reputation system** had a flaw that made submitted feedback viewable immediately, even if the other party had yet to leave their own feedback. This lead to “carrot and stick” kinds of concerns, and potentially to underreporting of negative experiences. Potential fixes investigated by eBay included simultaneous revealing of feedback, one-way detailed seller ratings, and only allowing positive feedback from sellers about buyers.

5.2 Exercise

1. Reputation
   
   (a) Comment on eBay’s old reputation system. What were some pros and cons of this system?
   
   (b) Discuss the pros and cons of a simultaneous-feedback system (i.e. the one eBay considered implementing).

   (c) Discuss the pros and cons of the detailed seller rating system (i.e. the one eBay implemented).

6 Information Elicitation

6.1 Review

1. Scoring rules attempt to incentivize agents to report their true beliefs about an outcome that will occur future. An agent’s payment (which can be negative) is a function of her report and
the realized outcome. A scoring rule is strictly proper if it incentivizes an agent to report her true belief.

2. The quadratic and logarithmic scoring rules are examples of strictly proper scoring rules. The logarithmic scoring rule is $t_{\log}(q,o_k) = \ln(q_k)$. The quadratic scoring rule is $t_{\text{quad}}(q,o_k) = 2q_k - \sum_{k' = 0}^{m-1} q_{k'}^2$ (for $m$ possible outcomes).

3. If $t(q,o_k)$ is a strictly proper scoring rule on $m$ outcomes, then a positive-affine transform $t'(q,o_k) = \alpha_k + \beta \cdot t(q,o_k)$, with $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}_{\geq 0}$, is strictly proper.

4. In the peer-prediction method, agents are asked to report their true signal, and payments are made based on reports of multiple agents. The agents report signals without knowing what the other agents’ reports are. Two peer-prediction mechanisms are output agreement and $1/prior$. Another mechanism uses strictly-proper scoring rules.

5. A practical concern is that of additional, uninformative equilibria. Participants in a peer-prediction mechanism may coordinate and receive high payments without revealing any useful information.

6.2 Exercise

Suppose the world has hidden states $H = \{0, 1\}$, with $P(H = 0) = 0.4$. Further suppose that there are binary signals $\{0, 1\}$, with state-conditional distributions $P(S_i = 0|H = 0) = 0.6$ and $P(S_i = 1|H = 1) = 0.8$.

1. Calculate the joint distribution on signals.

2. Calculate the payments in truthful equilibrium under Output Agreement. Is OA strictly proper here? Justify your answer.

3. Are there additional, uninformative equilibria under Output Agreement? If so, what are they?

4. Calculate payments under the logarithmic scoring rule for each pair of reports. Give transformed payments such that strict properness still holds but payments are between 0 and 1.

5. Calculate payments under the $1/prior$ peer prediction mechanism. Is this mechanism strictly proper?

7 Prediction Markets

7.1 Review

1. Prediction markets are systems for aggregating the beliefs of many people about one or more future events by allowing agents to place bets on possible outcomes. The challenge in designing
prediction markets is to create incentives such that those with new information or strongly held beliefs will choose to participate and trade in a way that reflects their information.

2. Agents can buy contracts on the outcome of an event. Ideally, if an agent believes the probability of an event to be \( p \), then he or she would buy contracts on the event at a price less than \( p \) and sell at a price greater than \( p \). Types of contracts include winner-take-all and index contracts (which pay out according to a quantifiable outcome, such as vote share).

3. A winner-take-all contract pays off if the underlying event occurs, and the contract owner gets nothing if the event does not occur. The current market price corresponds to the “market’s belief” regarding the probability of the event. If the contract pays off $10 when the event happens, and the contract is trading at $3, then the market believes that there is a 30% chance that the event will occur, assuming risk neutrality.

4. Two important designs are the continuous double auction (CDA) and the automated market maker (AMM). In the CDA, an order book maintains outstanding bids and asks. Orders are matched and carried out whenever there is a pair of a bids and asks such that the bid price is lower than the ask price. In the AMM, the market operator acts as a market maker and is always willing to trade. One main advantage of this design is greater liquidity.

5. For AMM, we want a mechanism with the following properties: no round-trip arbitrage, strictly positive prices, normalization, responsiveness, high liquidity, myopic incentives, and bounded loss. The log market scoring rule (LMSR) AMM is a market maker with these properties.

6. Prediction markets advantages over classical methods such as asking experts and opinion polls include high predictive accuracy, self-selection, incentive alignment, agents putting money behind their bets, running in real-time, and self-organizing. What are the limitations?

7.2 Exercise

7.2.1 Automated Market Maker with Market Scoring Rule

We analyze a market with an automated market maker using a logarithmic market scoring rule. Suppose that there are two outcomes, \( o_0 \) and \( o_1 \), possible, and that the automated market maker uses the cost function

\[
C(x_0, x_1) = 2 \ln(e^{x_0} + e^{x_1}).
\]

The initial state is \((0, 0)\).

1. A trader, \( A_0 \), puts in a buy order for 1 contract of \( o_0 \) occurring. What does he pay?

2. A trader, \( A_1 \), puts in a buy order for 2 contracts of \( o_0 \) occurring. What does he pay?

3. Compute the profits or losses of \( A_0 \) and \( A_1 \) if outcome \( o_0 \) occurs and if outcome \( o_1 \) occurs.

4. Suppose the market is currently trading at \( p = .40 \) for \( o_0 \). A new participant who believes that \( o_0 \) will occur with prob 0.8 comes along. Assuming she is risk neutral and not budget-constrained, how should she determine what trade to perform?