1 Game gadgets

1.1 Why do we love them?

Game gadgets are an important part in showing that End of the line reduces to Brouwer which reduces to Nash, therefore showing that Nash is PPAD-hard. If you’re really interested, see http://www.cs.yale.edu/homes/jf/ppad.pdf

1.2 Types of Game Gadgets

The gadgets enumerated in the paper are:

- equals
- addition
- subtraction
- multiplication
- threshold

Note that we use min(1, something) or max(0, something) in cases where the quantity we’re trying to compute has the possibility of going out of the [0,1] bound.

1.3 How do I construct these things?

1.3.1 4 agent gadgets:

For addition, subtraction, equals, and multiplication, we require 4 agents, with the third agent as an intermediary. The output agent always wants to do the opposite of what the third agent wants to do. If we don’t need a min or max, the third agent has the objective (in terms of actions!) as its reward for 0 and $a_4$ as its reward for 1. Then, when agent 3 is indifferent between 0 and 1, we have that the expected reward for playing 0 (the quantity we desire) is the same as the expectation of $x_4$.

For example, if the payment for playing 0 is $a_1 \ast a_2$ (multiplication gadget), then the expected value of playing 0 is $x_1 \ast x_2$ and the expected value of playing 1 is $x_4$. They must be equal in equilibrium, which is exactly what we want.
When the quantity could go out of bounds, just make sure to put the objective as the reward for agent 3 playing 0 and $a_4$ as its reward for 1. Now if the objective is less than 0, agent 3 always plays 1 (because nothing is better than negative), which means agent 4 always plays 0, achieving $\max(0, \text{objective})$.

If the objective is greater than 1, agent 3 always plays 0, so agent 4 always plays 1, achieving $\min(1, \text{objective})$.

### 1.3.2 2 and 3 agent gadgets:

Thresholding and scalar multiplication only involve one input and therefore need fewer agents.

For **scalar multiplication**, $\min(\alpha x_1, 1)$, we only need 3 gadgets, where the second gadget will be the mediator. Otherwise we proceed analogously. In this case, we have a min, so we set the objective, $\alpha a_1$ as the reward for playing 0 and $a_3$ as the reward for playing 1. Again, we can show that in expectation, this implies $x_3 = \alpha x_1$ when $x_1$ falls within $[0,1]$, and otherwise both agents 2 and 3 have a preference for actions 0 and 1, respectively.

For **thresholding**, we only need 2 gadgets. For a threshold of 1/2, simply set agent 2 to have a payoff of $a_1$ for 1 and $1 - a_1$ for 0. We can easily see that if $x_1 > .5$, we always play 1 and if $x_1 < .5$, we always play 0. We can adjust constants in the payoff to achieve a threshold we want, e.g. if we want to threshold $x_1$ at .9, let the payoff be $a_1 - .4$ for 1 and $1.4 - a_1$ for 0. We can see that if $x_1 > .9$, we prefer to play 1, etc.
1.4 Example Gadgets

1.5 How do I prove that the gadget is doing what I want?
Show that when the output is not equal to the desired behavior, we arrive at a contradiction. For the multiplication gadget, if $x_4 > x_1 \times x_2$, then agent 3 wants to play 1, but then agent 4 would play 0, which means 3 would have wanted to play 0, which is a contradiction.