CS 136 Assignment 1: Game Theory

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Out Saturday, Sept 14, 2018
Due 5pm sharp: Friday Sept 21, 2018
Submissions to Canvas class page

Total points: 79. This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of answers, and we encourage typed submissions. You are free to discuss the problem set with other students but you must not share your answers. Extra credit will only be considered as a factor in deciding the letter grade for the course at the end of the term.

1. [20 Points] Nash equilibrium

   (a) [3 Points] Prove Theorem 2.1. Be sure to argue both the necessary (“only if”) and sufficiency (“if”) directions.

   (b) [2 Points] What strategies survive iterated elimination of strictly-dominated strategies in this normal-form game?

   \[
   \begin{array}{ccc}
   & L & C & R \\
   T & 2,0 & 1,1 & 4,2 \\
   M & 3,4 & 1,2 & 2,3 \\
   B & 1,3 & 0,2 & 3,0 \\
   \end{array}
   \]

   (c) [4 Points] What are the pure strategy Nash equilibria of the game? Find a non-trivial (i.e., with support of two or more actions) mixed-strategy NE.

   (d) [5 Points] Prove that iterated elimination of strictly-dominated actions never removes an action that is in the support of any mixed-strategy Nash equilibrium. [Hint: proof by contradiction]

   (e) [1 Points] Give an example of a simple, simultaneous-move game where no action can be eliminated by iterated elimination of strictly-dominated actions.

   (f) [5 Points] Explain why the time complexity of iterated elimination of strictly-dominated actions by pure strategies (don’t worry about mixed strategies) is \(O(m^{n+2}n^2)\) for \(n\) players, each with \(m\) actions.

2. [12 Points] Scheduling game

   Consider a game that involves scheduling tasks onto shared machines. Each agent \(i \in \{1, 2, 3\}\) has a task to complete and chooses a machine. Let \(c_{ij}\) denote the time the task of agent \(i\) takes on machine \(j\).

   The costs are: \(c_{11} = 12, c_{12} = 10, c_{21} = 16, c_{22} = 10, c_{31} = 2, c_{32} = 16\), so that machine 2 is faster than machine 1 for jobs 1 and 2, but not for job 3. Each machine has a precedence
order, determining the sequence it will work on assigned tasks. The cost to an agent is the time when its own task completes, and an agent wants to minimize its cost.

Each machine adopts a shortest-first precedence order, preferring tasks that are shorter, and breaking ties in favor of agents with a lower index.

(a) [4 Points] What is the precedence order over the tasks for each machine? Give a pure-strategy Nash equilibrium, and argue why it is an equilibrium.

(b) [4 Points] Explain, without enumerating all possible action profiles, why the Nash equilibrium identified in part (a) is unique.

(c) [4 Points] The make-span is the latest time that any task is completed. What is the make-span in the Nash equilibrium? What is the socially optimal assignment; i.e., the one that minimizes the make-span? Justify your answer.


Consider a variation on the Bargaining game, in which the payoffs to the players after action me, even, and you from player 1, and “yes” from player 2, are (2,0), (1,1), and (0,2), respectively. The payoffs remain (0,0) if player 2 responds with “n” to an offer.

(a) [4 Points] Write out the normal-form representation, and determine the set of pure strategy Nash equilibria

(b) [3 Points] Determine the set of subgame-perfect equilibria. What is it about this variation on the Bargaining game that yields multiple subgame-perfect equilibria?

(c) [3 Points] Change the payoffs in response to “n” from player 2 (using the same payoffs after “n” everywhere in the game) so that even and y are played in the equilibrium play of the unique subgame-perfect equilibrium.

4. [14 Points] Infinitely repeated Prisoners’ Dilemma

(a) [10 Points] Prove that TtT is not a subgame-perfect equilibrium of the infinitely repeated Prisoners’ Dilemma when adopted by both players, for any discount factor \( \delta < 1 \). [Hint: use the single-deviation principle. There are four classes of strategically equivalent subgames, and you should show that there are two classes of subgames for which there is no discount factor that simultaneously precludes a profitable single-period deviation in both classes.]

(b) [2 Points] Establish the range of discount factor \( \delta \) for which the revised grim trigger in Figure 4.10 is a subgame-perfect equilibrium when adopted by both players. For this, follow the analysis approach in the proof of the Nash-Threat Folk theorem (Theorem 4.8).

(c) [2 Points] Give a brief, non-technical explanation for why the modification from the grim trigger in Figure 4.7 is important in obtaining a subgame-perfect equilibrium.

5. [9 Points] Correlated equilibria

(a) [3 Points] Explain why any distribution on pure-strategy Nash equilibria (e.g., play NE one with prob 1/3, and NE two with prob 2/3) is a correlated equilibrium of the same game.
(b) **[3 Points]** Give an example of a normal-form game (different from the one in Figure 2.6) that has a correlated equilibrium that is not simply a distribution on pure-strategy or mixed-strategy Nash equilibria. Justify your answer.

(c) **[3 Points]** Prove that the correlated equilibria described for Chicken in Example 2.7 (p. 29) are all *Pareto optimal* distributions on action profiles (see Section 2.1.3), for any signal distribution.

6. **[14 Points]** Succinct game representations

(a) **[2 Points]** How can a congestion game in which each agent’s action set is the set of singleton resources (e.g., $A_i = \{\{1\}, \{2\}, \{3\}\}$ for 3 resources) be represented as an action-graph game?

(b) **[3 Points]** Formulate deciding which dining hall to have dinner in as a congestion game, with each dining hall as a resource (you can go to any college house!). What are two problems with the model, and would this be better formulated in the agent-graph or action-graph representation (justify your answer, but no need to give this alternate formulation).

(c) **[3 Points]** Express matching pennies (Fig 2.3) as an action-graph game [Hint: see Fig. 2.11 b, and the idea of distinct actions per agent].

(d) **[3 Points]** Consider the agent-graph game in Figure 1. Suppose each agent has $m = 3$ actions. What is the exact number of entries in the payoff matrix for each agent? Explain why there is an action-graph representation with exactly the same representation size for agent utilities [Hint: Fig. 2.11 b, again!].

![Figure 1: A simple agent-graph game.](image)

(e) **[3 Points]** Explain the proof idea of Theorem 2.4, by providing simple summaries of the arguments made in the reading, and what you have explained in part (d). No need to be technical. [Hint: “exponentially more succinct” means (i) for any agent-graph game, it can always be represented as an action-graph game whose size is no more than polynomially larger than the agent-graph representation, AND (ii) there are games where the agent-graph representation is exponential in $n$ while the action-graph is not.]

(f) **(extra credit)** Explain why the game of Matching pennies (Figure 2.3) cannot be represented as a congestion game. [Note: a proof of this needs to consider representations that make use of multiple resources.]
7. General questions (Extra credit)

(a) Provide a precedence ordering for machines 1 and 2 in the scheduling game in Problem 2 for which there is no pure-strategy Nash equilibrium. Explain.

(b) Prove that Tft is a Nash equilibrium of the infinitely repeated Prisoners’ Dilemma when adopted by both players, for some discount factor \( \delta < 1 \). (Caution, Tft is not a subgame-perfect equilibrium, and so you will not be able to show this using the single-deviation principle.)

(c) Prove that the automaton strategy in Figure 4.11 forms a Nash equilibrium of the infinitely repeated Prisoners’ Dilemma when adopted by both players, for a suitably large \( \delta ( < 1 ) \), and with lexicographic preference for simplicity. Trace the behavior of this strategy when adopted by both players.