CS 136 Assignment 5
Revenue-Optimal Auctions

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Out Friday October 6, 2017
Due 5pm sharp: Fri. Oct. 13th, 2017
Latest possible submission: 5p, Sat Oct 14th. (solns distr. then)
Submissions to Canvas

Total points: 44. This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of answers, and we encourage typed submissions. You are free to discuss the problem set with other students but you must not share your answers. Extra credit will only be considered as a factor in deciding the letter grade for the course at the end of the term.

1. [14 Points] Prior-free auctions (digital goods)

(a) [2 Points] Construct an example of bids in the DOP auction for which a different price is offered to two different bidders. Why might this be undesirable?

(b) [2 Points] Show that the SPSB auction obtains an approximation factor of 2 to revenue target $R_{opt}^{(2)}(v)$ when there are two bidders.

(c) [4 Points] Consider a setting with two bidders ($v_1 \in [0,1]$), two goods, and deterministic, individually-rational, strategy-proof auctions. Prove that for any auction $A$, there is a value profile $(v_1, v_2)$ for which some other auction $B$ has strictly more revenue. [Hint: reasoning about agent-independent prices (Thm. 7.7) may be useful.]

(d) [6 Points] Prove that the profit extractor is strategy-proof. [Hint: monotonicity and critical value]

2. [14 Points] Random sampling auctions

The random optimal price (ROP) auction splits the bids into two sets $B_1$ and $B_2$. It then uses the DOP approach— bidders in $B_1$ each face the optimal price based on bids in $B_2$, and bidders in $B_2$ each face the optimal price based on the bids in $B_1$ (the optimal price is defined as per the definition in DOP). In the case that there are no bids in $B_1$ then the item is not sold to bids in $B_2$ (and similarly when there are no bids in $B_2$).

(a) [4 Points] Consider an instance with two bidders, one with value 1 and one with value 2. What is the revenue target $R_{opt}^{(2)}(v)$? Show that $R_{opt}^{(2)}(v)/E[R_{ROP}(v)] = 4$ on this instance. [Hint: to calculate the expected revenue of ROP, proceed by case analysis on the different outcomes of the random split.]
(b) [4 Points] What is the approximation factor achieved by the RSPE auction and the DOP auction on the same input?

(c) [6 Points] Write a simple program to simulate the ROP and RSPE auctions on the 10@10 and 90@1 example (see Example 9.7). How does the average revenue of the ROP and RSPE auctions compare to the DOP auction in this example? Give some intuition for what you find.

3. [19 Points] Virtual values

(a) [3 Points] Give an example of a piecewise-constant PDF for a distribution on values that is not regular.

(b) [4 Points] Calculate the expected virtual value in the SPSB auction with reserve $r = 0.5$, and two bidders with values IID $v_i \sim U(0,1)$. Compare with the expected revenue (see Example 9.1). What do you notice?

(c) [2 Points] Confirm that the virtual value (9.4) is correctly defined for this distribution function:

$$G(w) = \begin{cases} 2w & \text{if } w \leq 1/4 \\ \frac{2}{3}w + \frac{1}{3} & \text{otherwise} \end{cases}$$

(d) [2 Points] What is the expression for the virtual valuation function for a uniform distribution $v_i \sim U(v_\ell, v_h)$? Do you find anything surprising about this expression?

(e) [3 Points] Consider selling to a single bidder, and making a take-it-or-leave-it offer of price $p$. Recall Thm 9.2 (expected revenue = expected v.v.). Find the $p$ that maximizes expected v.v. when the bidder value is $v_i \sim U(v_\ell, v_h)$.

(f) [3 Points] Calculate the optimal price $p$ for uniform distributions on $[0,1]$, $[0.5, 1.5]$, $[1,2]$. What do you notice that is surprising, and can you provide a simple intuition for why this might make sense?

(g) [2 Points] What is the expected virtual value for an allocation rule that always allocates to a bidder with value $v_i \sim U(v_\ell, v_h)$? Give some intuition for this.

(h) (Extra Credit) Referring to Theorem 9.2:

(i) Changing the order of summation: Prove that $\sum_{i=1}^{k} \sum_{j=1}^{k} g(j)h(i) = \sum_{i=1}^{k} \sum_{j=1}^{i} g(i)h(j)$, for any $k \geq 1$ and any two functions $g$ and $h$.

(ii) Write down (without proof) the analogous identity for

$$\int_{w=0}^{\infty} \int_{z=w}^{\infty} g(z)x^*(w)dzdw.$$

Use this to confirm the ‘changing the order of integration’ step in the proof of Theorem 9.2.

(iii) Prove a version of Theorem 9.2 for the case of an auction with a single bidder, and assuming that the allocation rule $x(w) = 1$ for $w \geq r$ and 0 otherwise. [Hint: start with $E[\phi(w)x(w)] = \int_{w=r}^{\infty} \phi(w)g(w)dw$, and show for a sequence of steps that this is equal to $E[t(w)]$ for payment rule $t$.]