1. [12 Points] Bidding Languages

(a) [3 Points] Prove that the OR language is expressive for superadditive valuations and only superadditive valuations.

(b) [2 Points] The XOR-of-OR language allows a bidder to submit a bid such as \(((AB, 4) \lor (CD, 6)) \oplus ((EF, 10) \lor (FG, 12))\). How would this bid be represented in the OR* language?

(c) [4 Points] How many dummy items are needed to represent a general XOR-of-OR bid as an OR* bid? Assume \(k\) OR clauses, each with \(\ell\) atoms.

(d) [1 Points] Is the XOR-of-OR language fully expressive? Why or why not?

(e) [2 Points] Give an informal argument for why OR* bids “look just like” OR bids from the perspective of winner determination once dummy items are introduced into the supply. A consequence of this equivalence will be that we can use any algorithm for solving WDP with OR bids to solve WDP with OR* bids.

(f) [extra credit] A bidder with the majority valuation function values any package of size at least \(m/2\) items at $1, and any smaller package at $0. Show that the majority valuation function requires \(\binom{m}{m/2}\) atomic bids to represent in the OR* language. [Hint: it is helpful to argue that you do not need atoms smaller than \(m/2\).]

2. [5 Points] Winner determination

(a) [3 Points] Consider two bidders and three items, and XOR bids “\((A, 1) \oplus (BC, 2)\)” from bidder 1 and “\((AB, 2) \oplus (C, 3)\)” from bidder 2. Cyclic structure (Section 11.3.3) only applies to bids in the OR or OR* language (treating dummy items as items for the purpose of the structure). Explain why these bids cannot satisfy the cyclic structure property (Section 11.3.3) when expressed in the OR* language.
(b) [2 Points] Use one of the properties S1–S4 to explain why the winner determination problem for XS (XOR-of-Singletons) is tractable.

3. [10 Points] VCG auction, Core-selecting auctions

(a) [2 Points] Use (11.22) to confirm, by setting payments equal to value (and thus, bidders’ payoffs to zero), that the core of the CA is always non-empty.

Consider an instance with four goods \{A, B, C, D\}, and three bidders, with single-minded valuations \((A, 10), (ABCD, 19), (B, 8)\) respectively.

(b) [3 Points] What is the outcome of the VCG mechanism (with truthful bidding)? How can the losers collude and win and pay zero?

(c) [3 Points] A core outcome of an auction has the following properties, all defined with respect to reported valuations:

- outcome \(X\) is efficient
- no allocated bidder pays more than its value
- unallocated bidders pay zero
- Eq. (11.24) holds for all subsets of allocated bidders (including the set of all allocated bidders)

A core outcome is bidder-optimal if, in addition, the outcome satisfies the property of Defn. 11.8. Assuming truthful bidding, describe the bidder-optimal core outcome or set of bidder-optimal core outcomes on this input. Is the VCG outcome in the core?

(d) [2 Points] Now consider the deviation in which the losing bidders collude as in part (b). Describe the bidder-optimal core outcome or set of bidder-optimal core outcomes on this input. What do you notice?

4. [10 Points] Two-sided matching

(a) [2 Points] Verify that the matching selected by the teacher-proposing DA in Example 12.3 is (weakly) better for every teacher but (weakly) worse for every student than the matching selected by the student-proposing DA.

(b) [4 Points] Prove that there exists no mechanism for two-sided matching that is both stable and strategyproof. [Hint: consider the following preference orders for \(s_1, s_2, t_1, t_2\): \(t_1 \succ s_1, t_2 \succ s_2, t_2 \succ t_1, s_2 \succ t_1, s_1 \succ t_2, s_2 \succ s_1\). “Truncation” misreports are allowed.]

(c) [2 Points] Use the lattice property of stable matchings (end of Section 12.2.2) to prove that every student weakly-prefers any stable matching to the teacher-optimal matching.

(d) [2 Points] Show that if the student-optimal and teacher-optimal stable matchings are the same then there is a unique stable matching.

5. [11 Points] Public school choice

Suppose for simplicity that each school has capacity one (though most of our results will generalize to a many-to-one matching setting). Consider the following preference orders for students \(\{a_1, a_2, a_3\}\), and priority orders for schools \(\{b_1, b_2, b_3\}\). The priorities of schools may reflect a preference for under-represented minorities, or walk-zone students.
Use student-proposing DA to find the student-optimal, stable matching. Is there a matching (perhaps unstable) that Pareto dominates this matching for students?

(b) [4 Points] The Boston mechanism (used in Boston high schools until 2005) is defined as follows:

In step one, each student proposes to his or her first choice school, and students are matched with a school in order of school priority while there remains capacity. In each subsequent step \( k > 1 \): each unmatched student proposes to his or her \( k \)th most-preferred school, and students are matched with a school in order of school priority while there remains capacity. The mechanism terminates when all students are matched.

Run the Boston mechanism on the example in part (a). Show that student \( a_2 \) has a useful misreport. Give a general description of the kind of manipulation that can be useful in the Boston mechanism.

(c) [4 Points] A variation on the top-trading-cycles mechanism can be applied to public school choice:

In each step, each school with remaining capacity points to the unmatched student with highest priority, and each unmatched student points to his or her most-preferred school with remaining capacity. Paths alternate between students and schools, and “trading on a cycle” corresponds to each student on the cycle being matched with his or her requested school.

Run this mechanism on the example. Compare the stability and Pareto-optimality (for students) with the matching obtained by student-proposing DA. Can you interpret the mechanism as students trading priorities amongst themselves? By analogy to TTC, do you think the mechanism is strategy-proof for students (no need for a proof)?


(a) [4 Points] The following is an “edge formulation” of the problem of finding a maximum cardinality, vertex-disjoint matching. Interpret the variables, objective and constraints. Does the formulation limit cycle lengths? In terms of the number of pairs and compatibilities, how many decision variables and constraints are there in the edge formulation?

\[
\begin{align*}
\text{max} & \quad \sum_{(u,v) \in E} y_{uv} \\
\text{s.t.} & \quad \sum_{v \in V} y_{uv} \leq 1, \quad \forall u \in V \\
& \quad \sum_{v \in V} y_{uv} = \sum_{w \in V} y_{uw}, \quad \forall u \in V \quad (1) \\
& \quad y_{uv} \in \{0, 1\}, \quad \forall u, v \in V \\
\end{align*}
\]
(b) [2 Points] How does the size of the edge formulation compare to the size of the cycle formulation for cycles of length up to three (see bottom p. 313).

(c) [1 Points] What is a practical problem with the edge formulation when using the matching for kidney exchange?

(d) [4 Points] Figure 12.5 (c) shows an example of how an AB-O pair can be used on a 3-cycle to enable two pairs (O-A and A-AB) to match that would be unable to match otherwise. Provide examples of feasible 3-cycles that can form with each of the following pairs:

(i) AB-O (give one additional example)
(ii) AB-A (give one example)
(iii) B-O (give two examples)

extra credit Give an example of a 4-cycle that includes pair AB-O, and in which 3 pairs can match that would not otherwise be able to match.