CS 136 Assignment 8
Information Elicitation

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Out Friday November 9, 2018
Due 5pm sharp: Fri. November 16th, 2018

Total points: 49
This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of answers, and we encourage typed submissions. You are free to discuss the problem set with other students but you must not share your answers. Extra credit will only be considered as a factor in deciding the letter grade for the course at the end of the term. Submit to Canvas.

1. [10 Points] Scoring Rules
   (a) [3 Points] For binary outcome \{0, 1\}, plot the expected payment of the logarithmic scoring rule \( s_{\log} \) for different reports \( q_1 \in [0, 1] \) (the probability of outcome 1) given true belief \( p_1 = 0.8 \). Confirm from your plot that the rule is strictly proper.
   (b) [4 Points] Do the same thing for the quadratic scoring rule \( s_{\text{quad}} \). Describe two qualitative differences between the expected payments in the quadratic and logarithmic scoring rules.
   (c) [3 Points] An effective scoring rule is one where, given a restricted set of possible reports, such as \{0.1, 0.2, 0.4, 0.6, 0.97\}, an agent maximizes expected payment through making the report that is closest to its true belief. Do your plots suggest that one of these rules is effective, and one is not? An informal argument is fine.
   extracredit Can you think of a way to use a the quadratic scoring rule on binary signals to get a strictly proper scoring rule when the outcome space is continuous on [0,1]?

2. [14 Points] Peer prediction
   See Fig 17.4 for an example of a world-state model for correlated signals in peer prediction. For this question, assume hidden state \( H \) can take on value \{0, 1\}, and assume binary signals.
   Assume \( P(H = 0) = 0.6 \), and state-conditional probabilities \( P(X_i = 1 | H = 1) = 0.6 \) and \( P(X_i = 0 | H = 0) = 0.8 \).
   (a) [2 Points] What is \( P(X_2 = 1 \mid X_1 = 0) \) and \( P(X_2 = 1 \mid X_1 = 1) \), where \( X_1, X_2 \) are the random variables that represent the signals of agents 1 and 2, respectively?
   (b) [4 Points] Define the scoring-rule based, peer-prediction mechanism for this setting (section 17.2.3) using the quadratic scoring rule. Be sure to calculate the payment rule for each of agents 1 and 2 (for different reports \( r_1, r_2 \)), and also provide the 2 x 2 payoff matrix.
   (c) [3 Points] Verify that truthful reporting is a strict, correlated equilibrium of this mechanism.
   (d) [2 Points] Would the output agreement mechanism also be strictly proper for this signal distribution? Why or why not?
(e) [3 Points] Give three additional equilibria for the quadratic scoring rule based PP mechanism in this example. What challenge does this suggest with the practical application of peer prediction?

3. [25 Points] Prediction Markets

(a) [6 Points] CDA and Call Market
Consider a prediction market where contracts can be bought and sold that will pay out $1 if Boston breaks its all-time snow record in 2020. The following bids (B) and asks (A) arrive in each time period:

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<tr>
<th></th>
<th>B1</th>
<th>A2</th>
<th>A3</th>
<th>B4</th>
<th>A5</th>
<th>B6</th>
<th>B7</th>
<th>B8</th>
<th>A9</th>
<th>A10</th>
<th>A11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1</td>
<td>0.8</td>
<td>0.65</td>
<td>0.7</td>
<td>0.25</td>
<td>0.3</td>
<td>0.75</td>
<td>0.5</td>
<td>0.6</td>
<td>0.45</td>
<td>0.4</td>
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(i) [2 Points] What trades occur in a continuous double auction (recall the price is set by the matching order already in the order book, if any)
(ii) [2 Points] What trades occur in a call market in period 11, trading at a mid-point price?
(iii) [2 Points] Describe two qualitative differences between the outcomes in the two designs.

(b) [9 Points] Market Scoring Rules
Consider using a market scoring rule (see section 18.3.3) with the logarithmic proper scoring rule to elicit beliefs about the probability that Boston will break its all-time snow record in 2019.

There are three agents, with beliefs $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.8$, respectively (for the probability the record will be broken). Let $q^{(i)}$ denote the report of agent $i$, and assume the sequence of reports is 1, 2, then 3.

i) [2 Points] Explain why it is optimal to report truthful beliefs to a market scoring rule.
ii) [2 Points] Assume the initial belief $q^{(0)} = 0.5$. Give the expression for the expected payment of agent 1, as a function of its belief $p_1$ and its report $q^{(1)}$. Also calculate its payment when truthful, and in the event that Boston does not break the record.
iii) [2 Points] Calculate the total payment made to all agents by the scoring rule, in the event that the record falls, and assuming all agents report truthfully.
iv) [3 Points] What is the maximum total payment that the operator could possibly make, for any sequence of $k$ reports? Justify your answer.

(c) [10 Points] Automated Market Maker
The same three agents take turns trading with the LMSR automated market maker (section 18.3.2), with $\beta = 1$, so that the cost function in market state $x = (x_0, x_1)$ is

$$C(x_0, x_1) = \ln(e^{x_0} + e^{x_1}),$$

where contract 1 pays $1 when Boston breaks the record, and contract 0 pays $1 when the record does not fall. Suppose the market starts at market state $(x_0, x_1) = (0, 0)$.

i) [4 Points] Describe how the agents trade, assuming that agent 1, 2, and 3 make their trades in that order, and that each agent buys until the price is such that this is no longer profitable.
ii) [2 Points] Compute the profit to agent 1 in the event that the record does not fall, considering both the payment made to the market maker and payments received (if any) when the outcome is realized.

iii) [2 Points] Compute the total payments received by the market maker in the event that Boston breaks the record (consider payments made in settling contracts, as well as payments received in selling contracts).

iv) [2 Points] Compare the results with those in part (b). What do you notice? Suggest a simple explanation for why the automated market maker has a bounded loss, even if it was to sell an unbounded quantity of contract 1, and even in the event that the record falls and contract 1 pays out!