1 Game Theory

1.1 Normal Form Game

Generally, different players have conflicting interests, i.e. they want to maximize different objective functions. We use game theory to analyze such situations and make predictions given that each player seeks to maximize their utility given others’ actions.

A normal-form simultaneous-move game is defined by:

1. \( N = \{1, \ldots, n\} \) agents indexed by \( i \)

2. \( A = A_1 \times \ldots \times A_n \), where \( A_i \) is a set of actions available to agent \( i \) and \( a = (a_1, \ldots, a_n) \in A \) denotes an action profile

3. A payoff matrix or utility function \( u \) mapping each combination of actions to a utility for each agent

4. A mixed strategy \( s_i \) is a distribution on actions \( A_i \)

1.2 Solution Concept

A solution concept is a formal rule for predicting how a game will be played. Here are some basic solution concepts:

1. **Best response.** A strategy \( s_i \) is a best response to the strategies of other players if agent \( i \) maximizes its expected utility.

2. **Dominated Strategy:** Action \( a_i \) strictly dominates action \( a'_i \) if \( u_i(a_i, a_{-i}) > u_i(a'_i, a_{-i}) \) for all actions \( a_{-i} \) of others.

3. **Iterated Elimination of Dominated Strategies:** No player will ever choose a strictly dominated strategy, so we can iteratively eliminate strictly dominated strategies before solving for a Nash equilibrium.

4. **Nash Equilibrium (NE):** In a \( n \)-player game, strategy profile \( (s_1^*, \ldots, s_n^*) \) is a Nash equilibrium if \( u_i(s_i^*, s_{-i}^*) \geq u_i(s'_i, s_{-i}^*) \) \( \forall \) strategies \( s'_i \), for all players \( i \). In other words, every player’s strategy must be a best response to others. Nash equilibria can be pure strategy Nash equilibria (PSNE) (pure strategies only) or mixed strategy Nash equilibria (MSNE).
1.3 Congestion Game

A congestion game \((N, E, A, c)\) is defined by:

1. Set of players \(N = \{1, \ldots, n\}\)
2. Set of resources \(E = \{1, \ldots, m\}\)
3. Joint action set \(A = A_1 \times \cdots \times A_n\), where \(A_i \subseteq 2^E\) is the set of actions available to player \(i\)
4. Cost function \(c_e(x) : \{0, \ldots, n\} \rightarrow \mathbb{R}\) for resource \(e\) which depends only on the number of agents \(x\) that use resource \(e\)

Congestion games have the following properties:

1. Utility interpretation of cost: Given action profile \(a = (a_1, \ldots, a_n) \in A\) which leads to \(x_e\) players using resource \(e\) for each \(e \in E\), player \(i\)'s utility \(u_i(a) = -\sum_{e \in a_i} c_e(x_e)\).
2. Existence of pure strategy Nash equilibrium: Congestion games are potential games with potential function \(-\sum_{e \in E} \sum_{j=1}^{x_e} c_e(j)\) and therefore have a PSNE.

1.4 Sequential-Move Game

In sequential-move games, a player’s strategy needs to describe the action she will take after every sequence of observations about the actions of others:

1. Extensive form of sequential-move game: The extensive form is defined by
   
   (a) A set of players \(N = \{1, \ldots, n\}\)
   (b) A set \(H\) of action histories \(h\) (with empty history \(\varepsilon\))
   (c) A set \(Z \subset H\) of terminal histories and a utility \(u_i(h)\) for each player \(i\) and \(h \in Z\)
   (d) A mapping from each non-terminal history \(h \in H\setminus Z\) to a player \(i\) and action set \(A_i(h)\)

2. Strategies in extensive form: The strategy \(s_i\) in an extensive form game defines \(s_i(h) \in A_i(h)\) for every non-terminal history \(h \in H\setminus Z\) for which it is agent \(i\)’s turn to play. (Analogy: a player writes down her contingency plan for every situation before the game starts.)

3. Subgame-perfect Equilibrium: A strategy profile \(s^* = (s^*_1, \ldots, s^*_n)\) is a subgame-perfect equilibrium if it is a Nash equilibrium in the subgame at every non-terminal history \(h \in H\setminus Z\).

4. Single-deviation Principle: A strategy profile is a subgame-perfect equilibrium in a finite extensive form game \(iff\) there is no subgame at non-terminal history \(h\) where the agent whose turn it is to move has a useful deviation by changing his action at that history \(h\) only.
1.5 Repeated Game

In a repeated game $G^T$, the same simultaneous move stage game $G = (N, A, u)$ is played by the same players for $T$ periods, with every agent having perfect information about the history of actions. In an infinitely repeated game $G^\infty$, the stage game $G$ is repeated forever:

1. **Utility and discounting:** A player’s total utility is the sum of his utilities across stage games. In this sum, a player’s utility from game $k$ is discounted $\delta^k$, for $\delta \in [0, 1)$.

2. **Single-deviation Principle for Repeated Games:** A strategy profile is a subgame-perfect equilibrium in a (infinitely-) repeated game iff for every subgame, no agent can improve her utility by changing her action in the current period and leaving the rest of the strategy unchanged.

3. **Unique subgame-perfect equilibrium:** If the stage game has a unique Nash equilibrium, then a finitely repeated game has a unique subgame-perfect equilibrium, which is to play the stage game Nash equilibrium in every period.

4. **Open-loop strategy:** An open-loop strategy $s_i$ for player $i$ in a repeated game satisfies $s_i(h) = s_i(h')$ for all histories $h, h'$ of the same length. An open-loop strategy profile in which a stage game Nash equilibrium is played in each period is a subgame-perfect equilibrium of the repeated game.

5. **Automaton strategy:** An automaton strategy $m_i$ for player $i$ is defined by
   (a) A set $Q_i$ of machine states
   (b) A start state $q_0^i \in Q_i$
   (c) A next state function $q'_{i} = succ_i(q_i, a)$ for all states $q_i$ and action profiles $a \in A$
   (d) A mapping from states to actions $f_i : Q_i \rightarrow A_i$.

6. **Enforceable action** An action profile $a$ of the stage game is enforceable if there exists a (possibly mixed) Nash equilibrium strategy $s^*$ of the stage game, for which
   \[ \tilde{u}_i(a) > \tilde{u}_i(s^*) \]  
   for all agents $i$. $s^*$ can be thought as the Nash equilibrium where $i$ gets the lowest payoffs.

7. **Folk Theorem:** Given a stage game $G$ with an enforceable action profile $a$, there exists a subgame perfect equilibrium of the infinitely repeated game $G^\infty$, for all sufficiently large $\delta$, where action profile $a$ is played in equilibrium in every period.
   - **Cooperate:** Play according to $a$ if everyone played according to $a$ in all previous periods.
   - **Punish:** Otherwise, play strategy $s^*$ in every subsequent round.

Since repeating the Nash equilibrium play $s^*$ is an open loop strategy, it is SPE. Then if $i$ deviates, players can immediately go to the aforementioned Nash equilibrium to punish $i$ with the lowest possible payoff. (Note that the definition of Nash says that $i$ cannot do any better if he unilaterally deviates.)
1.6 Exercises

1. Three politicians are running in an election. Before the election, they participate in a pairwise debate. Every politician debates with each opponent separately, so that there are 3 debates in total. Before the debate series, each politician chooses an amount of negative campaigning \((a, b \text{ or } c)\), which cannot change during the election campaign. The outcomes of any pairwise debate are shown in the matrix below (you can think of them as the number of votes gained by the politicians as a result of a debate).

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>10,10</td>
<td>6,9</td>
<td>3,8</td>
</tr>
<tr>
<td>(b)</td>
<td>9,6</td>
<td>5,5</td>
<td>2,4</td>
</tr>
<tr>
<td>(c)</td>
<td>8,3</td>
<td>4,2</td>
<td>1,1</td>
</tr>
</tbody>
</table>

(a) Find all Nash equilibria of the game where the politicians care only about the total number of votes gained.

(b) Now suppose that politicians’ sole objective is to win the election, i.e. to get the largest total number of votes obtained in two debates. The tie-breaking rule for the election is randomization with equal probabilities. Is there a Nash equilibrium in which everyone plays the same strategy you found in part (a)? Find all Nash equilibria.

2. Consider the Battle of Sexes game as follows: Alice and Bob want to go out on a date. They would prefer to do things together, but have different tastes. Alice prefers opera while Bob prefers concert. Assume they agreed to meet this evening, but cannot recall which show they will be attending.

Here are their payoffs

<table>
<thead>
<tr>
<th>Alice \ Bob</th>
<th>Opera</th>
<th>Concert</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>(3, 2)</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>Concert</td>
<td>(0, 0)</td>
<td>(2, 3)</td>
</tr>
</tbody>
</table>

(a) Find the Nash Equilibria to the game.

(b) Show that either both going to the concert or both going to the opera is a sustainable on equilibrium path along some SPE in the finite repeated game.

2 Algorithmic Game Theory

2.1 Review

1. Two player zero sum games

   • Maximin Strategy: \(\tilde{s}_1 \in \arg\max_{s_1} \left[ \min_{a_2 \in A_2} u_1(s_1, a_2) \right]\)

\(^1\)Some exercises taken from Fudenberg’s Economics 1052: Game Theory
Minimax Strategy: $s_1 \in \arg \min_{s_1} \left[ \max_{a_2 \in A_2} u_2(s_1, a_2) \right]

Maximin and minimax value

Minimax Theorem: In any two-player, zero-sum game,
- For each player the set of maximin strategies is identical to the set of minimax strategies.
- Any maximin strategy for player 1 and any maximin strategy for player 2 is a NE, and these correspond to all NE’s.
- Maximin value = minimax value = expected utility.

2. Equilibrium of Two Player General Sum Games: This can be found using linear feasibility programs given a pair of candidate supports. Because one needs to check all possible sets of supports, and the number of possible supports is exponential in the number of actions, this algorithm has an exponential run time.

3. The problem of finding a NE (Nash) in General Sum Games is PPAD-Complete. This complexity class includes problems that always have a solution and where the solution can be found by walking on a potentially exponentially long path in a graph. It is likely that PPAD-hard problems cannot be solved in worst-case polynomial time.

4. Complexity classes:
   - P: yes-no decision problems that can be solved in worst case polynomial time, e.g. linear programming, 2SAT, max flow
   - NP: yes instances can be verified in polynomial time
   - NP-hard: “very hard” problems, not necessarily in NP, and at least as hard as the hardest in NP; e.g. the subset-sum problem
   - NP-complete: problems in NP and NP-hard (any problem in NP can be reduced to it in polynomial-time), e.g. 3SAT
   - PPAD: total search problems that can be reduced to End-of-the-Line.
   - PPAD-hard: problems to which End-of-the-Line can be reduced
   - PPAD-complete: problems in PPAD and PPAD-hard

5. Nash is PPAD-hard.

6. Correlated Equilibrium: Can be computed in worst-case polynomial time in the size of the payoff matrix. Unlike in n-player general sum game, where we are multiplying probabilities (and thus non-linear), the probability of a joint action is encoded as a single variable, so we are able to formulate linear feasibility programs.

2.2 Exercise

1. Explain how gadgets relate to algorithmic game theory.

2. Design a 4-agent gadget, where $x_1, x_2, x_4 \in [0, 1]$ are the mixed strategies of agents 1, 2 and 4, respectively, to implement $x_4 = \max(x_1 - x_2, 0)$. Agent 4 represents the output, with payoff 1 when it takes the opposite action to agent 3, and 0 otherwise. Agents 1 and 2 represent
the inputs, and have zero payoff for all actions of the other agents. Design the payoff for the middle agent 3, and prove (for any fixed \(x_1, x_2\)) that the required output property holds in every Nash equilibria.

3 Auction Theory

3.1 Review

1. Auctions: The possible goals of the designer:
   - Efficiency: allocate the item to the agent with the highest value
   - Revenue: Maximize expected revenue for the seller (design an optimal auction)

2. Value settings:
   - Private value: agent knows own value, value unaffected by value or information of others. Special case is Independent Private Values (IPV): independent and identically distributed (iid) valuations (e.g. uniform distribution)
   - Interdependent values: belief about an agent’s value may depend on value and information of others. Common value: item has same value to all, but uncertain (and beliefs about value may depend on value and information of others).

3. Quasi-linear utility and its role in auctions. [Value can be quantified in monetary units, thought of as “willingness to pay.”]

4. Dominant Strategy Equilibrium (DSE): A strategy profile \(s^* = (s_1^*, \ldots, s_n^*)\) is a DSE if, for all bidders \(i\)
   \[
   u_i(s_i^*(v_i), s_{-i}(v_{-i})) \geq u_i(b_i, s_{-i}(v_{-i}))
   \]
   for all \(v_i, b_i, v_{-i}, s_{-i}\) (2)

A DSE requires that the strategy for each bidder maximizes her utility, whatever the value and whatever the bids of others. Bidders do not need to reason about the values of others, or even believe that other bidders are rational.

5. Bayes-Nash equilibrium (applied to auctions):
   Strategy profile \(s^* = (s_1^*, \ldots, s_n^*)\) is a BNE in a sealed bid auction if for all agents \(i\) and values \(v_i\),
   \[
   E_{v_{-i}}[u_i(s_i^*(v_i), s_{-i}(v_{-i}))] \geq E_{v_{-i}}[u_i(b_i, s_{-i}^*(v_{-i}))], \text{ for all bids } b_i.
   \]
   (3)

6. Interim allocation, payments: the interim allocation \(x_i^*(v_i)\) is the probability of being allocated in equilibrium if value \(v_i\); the interim payment \(t_i^*(v_i)\) is the expected payment in equilibrium if value \(v_i\) (considering distribution on others’ values, allowing for both winning and losing the auction).
   - In any BNE of a normalized auction (smallest type have zero interim utility), the interim allocation is monotone non-decreasing in value, and the interim payment is
   \[
   t_i^*(v_i) = v_i x_i^*(v_i) - \int_{z=0}^{v_i} x_i^*(z) dz
   \]
The interim payment is completely pinned down by the interim allocation.

7. **Second-price sealed-bid auction**: The bidder with highest bid wins the item and pays the second-highest bid. Bidding true value is a dominant strategy. The auction is efficient in the truthful equilibrium.

8. **First-price sealed-bid auction**: The bidder with highest bid wins the item and pays her bid value. The BNE in IPV with uniform $U(0,1)$ values is $s^*(v_i) = \frac{n-1}{n}v_i$, where $n$ is the number of agents. Strategy symmetric and strict increasing, so auction is efficient.

9. **Revenue Equivalence Theorem**: all auctions with the same interim allocation have the same expected revenue (in equilibrium); e.g., FPSB and SPSB (in truthful equil.) have same expected revenue. This characterization can also be used for computing equilibria of non-standard auctions.

10. **Ebay auction design**

3.2 **Exercise**

Consider a first-price sealed bid auction, where each bidder $i$ has value $v_i$, sampled IID from $U(0,1)$. Depart from the quasi-linear model and assume instead that if bidder $i$ wins with bid $b_i$, her utility is $(v_i - b_i)^{1/m}$ where $m > 1$.

1. Show there is a symmetric pure strategy Bayesian-Nash equilibrium in which each bidder uses the strategy $s^*_i(v_i) = (1 - \frac{1}{m(n-1)+1})v_i$.

2. How does the seller’s expected revenue from this auction compare to the symmetric equilibrium of the FPSB auction with the same distribution of values, and with the standard quasi-linear utility of $v_i - b_i$?

3. How do the expected revenues from the FPSB and the SPSB auction compare when the utility functions of the bidders are as in part (a)?

4 **Peer-to-Peer Systems**

4.1 **Review**

BitTorrent: Torrent files contain metadata for a file, and the IP of a tracker. That tracker holds information about the swarm, the group of users currently downloading/uploading that file. Users can request a list of roughly 50 peers from the tracker, to begin downloading the file.

Peers exchange information with one another about which pieces of the file they have. Each person will download from any peers that will upload to them. But, they will only upload to those peers that meet certain conditions.

In the reference client, there are four upload slots. The first 3 slots are given to the peers from whom a peer has the highest download rate. The 4th is for ‘optimistic unchoking’, and given to a random peer with the hope that the peer will reciprocate.

Exploits: under-reporting pieces available for upload, strategic unchoking and maximize “return on investment”, uploading garbage (effectively prevented by hashing blocks), growing neighborhood quickly and free-riding (effectively prevented by having trackers limit multiple requests from the same IP).
4.2 Exercise
1. Explain why breaking files into small pieces is helpful.
2. How might you measure the performance of a swarm? Of a client protocol?
3. Why is optimistic-unchoking helpful for the system?
4. What does cooperation mean, and is this desirable for the system?

5 Mechanism Design
5.1 Key Concepts
1. Mechanism Design - where a system designer designs a mechanism which takes reports from self-interested agents (e.g. bids in an auction) and selects outcomes (i.e. allocation and payment) based on the agents’ reports. Some desirable characteristics: truthfulness and allocative efficiency.

2. Direct-revelation Mechanism (DRM)

   - (Mechanism Design without Money) : DRM is specified by an outcome rule \( g: U \rightarrow O \)
   - (Mechanism Design with Money) : DRM is specified by a choice rule \( x: V \rightarrow A \) and a payment rule \( t: V \rightarrow \mathbb{R}^n \)
   - Strategy of agent \( i, s_i: V_i \rightarrow V_i \).
   - A mechanism is strategyproof if it is a DRM and truthful reporting is a dominant strategy equilibrium (DSE).

3. Revelation Principle: Any function \( f \) that can be implemented in a DSE of an indirect mechanism, can also be implemented by a strategyproof mechanism.

4. VCG Mechanism : maximizes the total value of the agents and charges each agent the negative externality it imposes on the other agents by its presence. Formally, given reported valuation function \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \), the VCG mechanism is defined by
   - A choice rule \( x(\hat{v}) \in \arg\max_{a \in A} \sum_{i \in N} \hat{v}_i(a) \)
   - A payment rule \( t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(a^{-i}) - \sum_{j \neq i} \hat{v}_j(a^*) \) \( \forall \) agents \( i \) where \( a^* = x(\hat{v}) \) and \( a^{-i} \in \arg\max_{a \in A^{-i}} \sum_{j \neq i} \hat{v}_j(a) \).
   - The VCG mechanism is truthful, efficient and satisfies individual rationality.
   - The VCG mechanism sometimes runs at a deficit. No positive externalities ensures no deficit.

5. Single-parameter domains: in a single parameter domain, an agent’s valuation function is defined by a single number. In these domains, truthfulness is attained through the design of monotonic choice rules, so that increasing a user’s value can only shift the selected alternative to make it more favorable for the user. Given this, the payment identity gives the payment rule, and reduces to the critical-value payment (smallest reported value such that she would
get the same allocation, or same alternative would be selected.) Greedy algorithms tend to be monotonic. Examples include single-item auctions, TV advertising, committee assignment (binary domain).

6. Taxation Principle: a mechanism is strategy-proof if and only if there exists a family of pricing functions such that the choice rule is agent-optimizing and the payment is agent independent.

5.2 Exercises

1. Revelation principle: State the truthful (dominant strategy incentive compatible) DRM for the FPSB auction and values that are IID uniform on [0,1].

2. Consider the following buying-a-path-in-a-network problem. Each edge corresponds to an agent \( i \), with private cost \( c_i > 0 \) if the edge is used. We want to find the lowest cost path from node 1 to node 5.

(a) Run the VCG mechanism on this example and report the allocation and payments in this context.

(b) Does the VCG mechanism satisfy individual rationality on this example? Why or why not? What about no-deficit?