1 Mechanism Design

1.1 Review

Key Points to Understand

1. Single Parameter Domain: each agent has a private input $w_i \in \mathbb{R}$, and a summarization function $q_i : A \rightarrow \mathbb{R}_{\geq 0}$ (known to the mechanism designer). For an allocation $a$, agent $i$’s valuation function is $v_i(w_i, a) = w_i \cdot q_i(a)$. Examples include single item (possibly multi-unit) auction. A mechanism $(x, t)$ is strategy-proof (SP) for a single parameter domain if and only if, for all $i$ and $w_i$, $q_i(x(w_i, w_i)) = q_i(x(w'_i, w_i))$ for all $w'_i > w_i$.

- (monotonicity) A choice rule $x$ is **monotone non-decreasing**, if, for all $i \in N$, all $\hat{w}_i$, $q_i(x(w_i, \hat{w}_i)) \geq q_i(x(w'_i, \hat{w}_i))$ for all $w'_i > w_i$.

- (payment identity) payment rule $t$ satisfies

$$t_i(\hat{w}_i, \hat{w}_i) = \hat{w}_i \cdot q_i(x(\hat{w}_i, \hat{w}_i)) - \int_{z=0}^{\hat{w}_i} q_i(x(z, \hat{w}_i))dz$$  \hspace{1cm} (1)

2. Knapsack: we have some number $m \geq 1$ of identical, indivisible items to sell to a set of $n$ bidders. Each bidder $i$ demands a public quantity $k_i$ of items, but her private value $w_i$ on her bid is private; so, we have a report $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_n)$ of items. The **knapsack auction** is defined by the following allocation and payment rules:

- The allocation rule is as follows: first let $W_g$ be the value of the greedy allocation $N_{approx}$: remember that this ranks bidders in decreasing order of bang-for-buck, $\hat{w}_i/k_i$, and allocates in decreasing order, skipping a bid if it exceeds capacity. Let $W_h = \max_i \hat{w}_i$ be the value of the largest bid (assuming all $k_i$ are less than capacity). If $W_g \geq W_h$ items are allocated according to $N_{approx}$, else the highest bidder wins.

- The payment rule: simply the critical value (from single-parameter domains). In other words, 0 if a bidder is unallocated, else the minimum amount she must bid to be allocated.

3. Single-minded CA: there are bidders $N = \{1, \ldots, n\}$, a set $G$ of distinct and indivisible items, bidders report pairs $(\hat{T}_i, \hat{w}_i)$ (with true valuations $(T_i, w_i)$), with $T_i, \hat{T}_i \subseteq G$, and $w_i, \hat{w}_i \geq 0$. If bidder $i$ is allocated set $S$, then $v_i(S)$ is 0 if $T_i$ is not contained in $S$, else $w_i$. Suppose that $\sigma : 2^G \times \mathbb{R} \rightarrow \mathbb{R}$ is a monotone scoring function (so that $\sigma(w'_i, T'_i) \geq \sigma(w_i, T_i)$, if either $w'_i \geq w_i$ or $T'_i \subseteq T_i$ (or both)). The single-minded CA is defined by:
• Allocation rule: sort bids in order of decreasing score; accept greedily, skipping a bid if one or more items in the set $\hat{T}_i$ has been allocated.

• Payment: bidder $i$ is allocated by its critical value.

4. Min-makespan scheduling: $G$ is a set of tasks, $|G| = m$, and $c_{ij} > 0$ is the time agent $i \in N$ takes to complete task $j \in G$. Let $z \in Z$ denote a feasible assignment of tasks. The design objective is to minimize $\text{makespan}(z)$ over all assignments $z$, defined by

$$\text{makespan}(z) = \max_{i \in N} \left[ \sum_{j \in z_i} c_{ij} \right].$$

Theorem: for 2 agents, no deterministic mechanism can achieve an approximation ratio of better than 2 for the general problem of min-makespan scheduling.

So, we focus on the special case of a single-parameter setting: each agent has a type $w_i = -r_i$, the negative unit processing time. $v_i(w_i, z) = -r_i \cdot (\sum_{j \in z_i} t_j) = -r_i \cdot q_i(z)$, where $q_i(z)$ is the total amount of work provided to agent $i$. For the single-parameter problem, there is a deterministic SP optimal min-makespan mechanism (pset!).

5. Agent-independent pricing function: for every $v_{-i}$ there does not exist two different reports $v_i, v'_i$ of agent $i$ such that the same outcome is selected but $t_i(v_i, v_{-i}) \neq t_i(v'_i, v_{-i})$. In other words, for all $v_{-i}$, the payment is independent of $v_i$ as long as the same outcome is chosen. This together with “agent optimizing” (agents get what they want when facing these prices) are necessary and sufficient for SP.

1.2 Practice

1. Double Auction. There is a seller (1) with an item and a buyer (2). The outcome is trade or no-trade as well as payments by (or to) each agent. Each agent $i$ has private information $w_i$. Seller’s value for trade is $w_1 < 0$. Buyer’s value is $w_2 > 0$. Trade occurs if and only if $w_1 + w_2 \geq 0$. E.g., seller -5, buyer 10, then trade.

(a) Explain why the allocation rule is monotonic.

(b) Write down the payment rule corresponding to the payment identity.

(c) Verify that the payment identity gives the VCG payments.

Answer: (a) Higher bid from 2, then trade still happens. Less negative number (smaller ask) from seller, then trade still happens.

(b) Consider $i = 2$ first. $q_i(x(w_1, w_2)) = 0$ for as long as $w_2 < -w_1$ and payment 0. For $w_2 \geq -w_1$,

$$w_2 \cdot q_2(x(w_1, w_2)) - \int_0^{w_2} q_2(x(w_1, z))dz = w_2 - (w_2 - (-w_1)) = -w_1,$$

so the buyer pays $-w_1$.

1 Technically this domain is not single-parameter, but it is shown in the reading that the knapsack auction is SP with this payment rule.
Now consider \( i = 1 \). \( q_i(x(w_1, w_2)) = 0 \) for as long as \( w_2 < -w_1 \) and payment 0. For \( w_1 \geq -w_2 \), the payment is

\[
    w_1 \cdot q_1(x(w_1, w_2)) - \int_{-\infty}^{w_1} q_1(x(z, w_2)) dz = -w_1 - (-w_1 - (-w_2)) = -w_2,
\]

so the seller receives \( w_2 \) in payment.

(c) For no trade, VCG payments zero. With trade, if either the seller or the buyer not there, the value of the other person would be 0 (since no trade!). So, seller pays \(-w_2\) (and gets paid \( w_2! \)) and the buyer pays \( w_2 \).

2. **Binary domain** A sealed-bid auction for a single item (and values on 0, 1) is defined in terms of an allocation rule \( x : [0,1]^n \rightarrow \{0,1\}^n \) and payment rule \( t : [0,1]^n \rightarrow \mathbb{R}^n \). For a given valuation profile \( v \), \( x_i(v) = 1 \) if agent \( i \) gets the item and it is 0 otherwise. Recall the notions of *critical value payment* (an allocated bidder is charged the minimum bid so that she is allocated the item) and *monotone allocation* (the choice rule \( x_i \) is monotone non-decreasing in an agent’s report \( \hat{v}_i \)).

(a) Prove that any auction that has a monotone allocation rule and charges critical value payments has a dominant-strategy equilibrium where agents report truthfully.

(b) Define the second-price Vickrey auction in these terms (i.e. show the choice rule is monotone and that bidders are charged their critical value.)

**Answer**:

(a) Once agents receive the item, changing their valuation won’t change their payment since they are always charged their critical value. If an agent does not receive the item, then they must bid more to receive the item, and they will be charged that amount, which is greater than their true value.

For a more formal argument: suppose otherwise, i.e. there is some \( v_{-i} \) and \( v_i' \neq v_i \) for which agent \( i \) benefits by reporting \( v_i' \) instead of \( v_i \). In order for \( i \) to benefit, since \( i \) pays the same amount (the critical value) if she is ever allocated the item, she must not receive the item when reporting \( v_i \) but receive it when reporting \( v_i' \). But then

\[
    z' \cdot v_i - c_i(v_{-i}) > z \cdot v_i = 0.
\]

where \( z = x_i(v_i, v_{-i}) = 0 \) and \( z' = x_i(v_i', v_{-i}) = 1 \). But since \( i \) is not allocated when the other agents report \( v_{-i} \), and \( i \) reports \( v_i \), we must have that \( c_i(v_{-i}) > v_i \). But the above equation states that \( v_i > c_i(v_{-i}) \).

(b) For a second-price Vickrey auction, in order to win a bidder must pay the price of the second-highest bidder, which is the minimum amount they must bid to win. It is clearly monotone.

3. **VCG-based and single-minded CA** Using the below example, show that a VCG-based mechanism that uses the greedy algorithm for the single-minded CA is not strategy-proof. Show that all bidders have a useful manipulation. What do you notice about the effects of the manipulation on the allocation?
The example is: there are 3 items $A, B, C$, and 3 bidders with target valuations $(AB, 3), (AC, 2)(B, 2)$. The value-based scoring function $\sigma(\hat{T}_i, \hat{w}_i) = \hat{w}_i$ is used.

**Answer:** The greedy algorithm allocates bid 1, obtaining total value 3. Bidder 1 pays $2 + 2 - 0 = 4$, which is greater than her value, 3. Therefore, VCG is not even IR here! (In particular, bidder 1 could deviate and bid 0, obtaining total value $0 > -1$. Another deviation: suppose bidder 2 deviates and reports $(AC, 4)$. Then the allocation will switch to bidders 2 and 3 winning, and bidder 2 will pay $3 - 2 = 1$, which is less than her value of 2. Finally, bidder 3 can deviate an report $(B, 4)$; the allocation will switch to bidders 2 and 3 winning, and bidder 3 will pay $3 - 2 = 1$, which is less than her value of 2.

4. **Knapsack auction** Recall that in the knapsack auction the allocation is made by the greedy allocation if $W_g \geq W_h$, else the bidder with highest bid ($W_h$) is allocated.

(a) Find an example of an instance in which the total value by greedy allocation $W_g$ is greater than the largest bid value $W_h$. Check that no agent has a useful deviation.

**Answer:** (a) Suppose there are 2 items and 2 bidders whose types are $(3, 1), (3, 1)$. Here $W_g = 6 > 3 = W_h$. Both bidders are allocated and both bidders pay 0. (So clearly there are no useful deviation.)

(b) Suppose there are 3 items and 3 bidders whose types are $(3, 1), (3, 1), (7, 3)$. Here $W_g = 6 < 7 = W_h$. In order to be allocated bidders 1 or 2 would have to bid at least 4, and they would be charged 4, which is greater than their valuation. Bidder 3 will be charged 6 as long as she bids at least 6, so has no useful deviation.

5. **Single-parameter min-makespan** Suppose there are 3 agents in a min-makespan scheduling problem, with unit-processing costs $r_1 = 1, r_2 = 2, r_3 = 4$, and 3 tasks $l_A = 2, l_B = 3, l_C = 4$. (a) What are the implied cost functions? (b) What is the min-makespan assignment? What is the minimum makespan?

**Answer:** (a) 

<table>
<thead>
<tr>
<th>taskA</th>
<th>taskB</th>
<th>taskC</th>
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<tbody>
<tr>
<td>ag1</td>
<td>2</td>
<td>3</td>
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<tr>
<td>ag2</td>
<td>4</td>
<td>6</td>
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<tr>
<td>ag3</td>
<td>8</td>
<td>12</td>
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(b) $(AC, B, \emptyset)$, with makespan $\max(6, 6, 0) = 6$.

6. **LexOpt** The LexOpt scheduling rule for the single-parameter makespan scheduling problem is defined as follows: if $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_n)$ is the report profile, then $x(\hat{w})$ selects a minimum-makespan assignment, breaking ties lexicographically: assignment $z$ is preferred over $z'$ if there exists an agent $k$ such that $q_k(z) < q_k(z')$ and $q_i(z) = q_i(z')$ for all $i < k$.

(a) For the example in the previous question, what is the allocation by LexOpt? If $z$ denotes the allocation, then what is $q_1(z), q_2(z), q_3(z)$?

(b) Verify that if agent 1 reports $\hat{r}_1 < 1$, then the amount of work she is allocated $(q_1(z'))$ will never decrease.
What is the allocation of LexOpt if instead $r_2 = 4.5$ and $r_3 = 8$?

**Answer:**
(a) It is the same as the allocation in the previous example (there are no ties to break); in particular, $q_1 = q_2 = 6, q_3 = 0$.

(b) The allocation to agent 1 will stay the same as long as $\hat{r}_1 \geq 4/7$. For $4/7 > \hat{r}_1 \geq 4/9$, the allocation will be $(BC, A)$, and agent 1 will receive 7 units of work (which is more than 6). For $\hat{r}_1 < 4/9$ agent 1 will receive 9 units of work.

(c) In this case there are 2 allocations with minimal makespan of 9, namely $(ABC, \emptyset, \emptyset), (BC, A, \emptyset)$. LexOpt will choose the second one since agent 1 is assigned less work.

### 1.3 More practice problems

1. (a) Take the buying-a-path-in-a-network problem, shown in the figure. Each edge corresponds to an agent $i$ with cost $c_i > 0$ if the edge is used. Bob wants to get from $S$ to $F$ so he asks for path costs of all the agents and runs the VCG mechanism. Which path does Bob purchase, and what are the VCG payments for the agents along the path?

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(b) Bob uses the VCG mechanism to go from $S$ to $F$. But now wants to travel back home, from $F$ to $S$. He could use VCG again, but he can actually do better. What are some other path-procurement mechanisms that are better for Bob than using VCG? [Hint: now he knows more!]

**Answer:**
(a) With VCG, we find the cheapest path and then calculate payments along this path. The cheapest path is d-e, with a total cost to the owners of 35.

Now we calculate payments. Without d, the cheapest path is g-f-e, which costs 40. So the edge d pays $(40) - (-15) = -25$. In other words, edge d is compensated 25.

Without e, the cheapest path is a-b. This costs 50. So e pays $(50) - (-20) = -30$. In other words, e is compensated 30.
In total, Bob pays 55 and travels along d and e.

(b) Bob just ran the VCG mechanism, so he knows the true costs of all the edges. He could certainly run VCG again and pay 55, but perhaps Bob could negotiate prices with a and b. Bob could offer to pay a 21 and b 31, giving him a total cost of 52, which is better than the cost of 55 he paid through VCG. Perhaps Bob could even negotiate lower prices with d and e. Bob can offer to pay d 21 and e 16 and threaten to use path a-b in case d or e rejects the prices. This incurs Bob an even lower cost of 37.

Bob was not able to implement this negotiation in part (a) because he did not know the costs of any paths.

2. TV Advertising

Consider an auction where advertisers are bidding for Superbowl commercials. There are spots for three commercials to air and the iser with the first spot gets 40 views, the second gets 25 views, and the third gets 10 views (numbers in millions). Each advertiser wants one of the three commercials and only cares about the number of views she gets.

(a) Assume a mechanism designer runs an auction where bidders bid their willingness to pay per view. Come up with a choice rule that is monotone non-decreasing in every report \( \hat{v}_i \).

(b) Consider the simple monotone non-decreasing choice and payment rules where the highest bidder receives the top spot and pays the second highest price per view; the second highest bidder receives the second spot and pays the third highest price per view, and the third highest bidder receives the third spot and pays the fourth highest price per view. Is this mechanism strategyproof? If so, show that it is strategyproof, if not come up with an example where it is not strategyproof.

Answer: (a) The highest bidder receives the top spot, the second highest receives the second spot, and the third highest receives the third spot. Break ties by awarding the higher spot to the agent with the lower index.

(b) No. This mechanism is not strategyproof. Imagine a situation where four agents have valuations per view of 20, 19, 2, and 1. Under this mechanism, the player who bids 20 pays 19 on 40 views, giving him a total utility of \((20 - 19) \times 40 = 40\). However, if instead he were to bid 18, he would come in second, pay 2 and receive 25 views, giving him a total utility of \((20 - 2) \times 25 = 450\).

(c) First note that, given a valuation profile \( v \) of the four agents, the allocation \( x(v) \) is a four-dimensional vector where \( x_i(v) \) is the slot allocated to ad \( i \). For example, \( x_4(v) = 1 \) if ad 4 gets the first spot, \( 2 \) if she gets the second spot, \( 3 \) if she gets the third spot and \( 0 \) otherwise. Since the advertiser is only interested in number of views of an allocation, the summarization
function maps the allocation to the corresponding number of views i.e.

\[
q_4(x(v)) = \begin{cases} 
40 & \text{if } x_4(v) = 1 \\
25 & \text{if } x_4(v) = 2 \\
10 & \text{if } x_4(v) = 3 \\
0 & \text{if } x_4(v) = 0 
\end{cases}
\]

Now, we want to use the payment identity. To start thinking about this, we draw a graph with the summarization function \(q_4(x(v_4, v_{-4}))\) along the y-axis and \(v_4\) of our fourth agent along the x-axis.

The payment identity gives us

\[
t_4(v_4, \langle 4, 6, 10 \rangle) = v_4 \cdot q_4(x(z, v_{-4})) - \int_{z=0}^{z=v_4} q_4(x(z, v_{-4}))dz.
\]

For a given value of \(v_4\), the agent’s utility is \(v_4 \cdot q_4(x(v_4, v_{-4})) - t_4(v_4, \langle 4, 6, 10 \rangle)\) which is nothing but the area under the allocation curve from 0 to \(v_4\). The shaded area in the next graph shows the agent’s utility if his valuation is 12.
From the graph we can express utility of agent 4 as a function of $v_4$.

$$u_4(v_4, v_{-4}) = \begin{cases} 
0 & \text{if } 0 \leq v_4 \leq 4 \\
10(v_4 - 4) & \text{if } 4 < v_4 \leq 6 \\
20 + 25(v_4 - 6) & \text{if } 6 < v_4 \leq 10 \\
120 + 40(v_4 - 10) & \text{if } v_4 > 10 
\end{cases}$$

The below graph illustrates his utility plotted against all values of $v_4$. 

![Utility graph](image-url)