1 Review of Revenue Optimal Auctions

1. The virtual value function for bidder $i$ with value distribution $G_i$ (and density $g_i$) is
\[
\phi_i(v_i) = v_i - \frac{1 - G_i(v_i)}{g_i(v_i)}.
\]
A value distribution $G_i$ is regular if the virtual value function is strictly increasing.

2. Recall that a normalized auction is BIC iff for all bidders $i$, we have:
   (a) The bidder’s interim allocation $x^*_i(w)$ is monotone non-decreasing in the bidder’s value $w$ (interim monotonicity), and
   (b) The bidder’s interim payment is $t^*_i(w) = w \cdot x^*_i(w) - \int_0^w x^*_i(w)dw$, for all values $w$, and where $x^*_i(w)$ is the interim allocation (payment identity).

3. The expected revenue collected from a single bidder in a normalized BIC auction is equal to the expected virtual value of the bidder:
\[
E_w[t^*_i(w)] = E_w[\phi_i(w)x^*_i(w)].
\]
This is proved by integrating the payment identity from the above bullet over $w$ and using the definition of virtual value. If you ever forget what the virtual value function is, going through this derivation is a good way to remind yourself of it.

4. It follows as a direct consequence of the above result that the expected revenue in a normalized BIC auction with allocation rule $x$ is equal to the virtual value of the allocation, namely $E_{v_1,...,v_n}[\sum_{i \in N} \phi_i(v_i)x_i(v)]$.

5. (Myerson auction for regular value distributions). Suppose that bids are ordered in increasing order of virtual value, so that $\phi_1(b_1) \geq \phi_2(b_2) \geq \cdots \geq \phi_n(b_n)$, for virtual value function $\phi_i$. The Myerson auction allocates to bidder 1 if $\phi_1(b_1) \geq 0$, and collects as payment $t_{Mye,1}(b) = c$, where $c \in [0, v_{max}]$ is the smallest value such that $\phi_1(c) \geq \max(\phi_2(b_2), 0)$. If $\phi_1(b_1) < 0$, then the item is left unallocated. Unallocated bidders pay 0. Using the above theorems, it can be shown that if value distributions are regular, then the Myerson auction is strategyproof and maximizes expected revenue.

6. If bidders’ distributions are iid (not just independent) with common virtual value function $\phi$, then the Myerson auction corresponds to the SPSB auction with reserve price $\phi^{-1}(0)$.
2 Review of Prior-free auctions

1. We are in the digital goods setting, where we can sell as many goods as we want. We want a constant approximation factor to the strategyproof auction which “maximizes revenue” for any given set of bids. The big question, though, is what exactly does “maximizes revenue” mean in this setting? To specify this, we must give a revenue target.

2. An auction is a \(c\)-approximation for revenue with respect to a revenue target if it comes within a fraction \(c\) of the target for all possible inputs. That is,

\[
\frac{R_{\text{opt}}(v)}{R_{\text{approx}}(v)} \leq c
\]

for all \(v\), where \(c \geq 1\). A lower bound \(\beta \geq 1\) is a proof that no auction can achieve a \(c\)-approximation for any approximation factor \(c < \beta\). If we find an auction that gives a \(c'\) approximation, for some \(c' \geq 1\), then \(c'\) can be viewed as an upper bound on the minimum approximation ratio possible.

3. One possible revenue target is the following: if values are ordered \(v_1 \geq \cdots \geq v_n\), then define \(R_{\text{opt}}^{(1)} = \max_{i \in \{1, \ldots, n\}} i \cdot v_i\); this is the maximum revenue that can be achieved if we offer one unit of the item to each bidder at the same price. Unfortunately, there is a distribution of a bidder such that there is no constant strategyproof approximation to the revenue target \(R_{\text{opt}}^{(1)}\) (even if we do not impose prior-freeness!).

4. Another possible revenue target is, \(R_{\text{opt}}^{(2)} = \max_{2 \leq i \leq n} i \cdot v_i\). This is the optimal revenue when selling at a single price to all bidders, and specifying that at least 2 items are sold.

5. One strategyproof prior-free auction we might consider is the DOP auction: given \(b = (b_1, \ldots, b_n)\), and an unlimited number of items for sale, the DOP auction calculates the optimal price \(p_i = p^*(b_{-i})\), for each \(i \in N\), and sells to \(i\) if and only if \(b_i \geq p_i\), at price \(p_i\). (Remember that the optimal price \(p^*(b)\) corresponding to a bid profile \(b\) is the bid \(b_j\) corresponding to \(j = \arg \max_{i \in N} i \cdot b_i\).)

6. The DOP auction does not provide a constant-factor approximation, but the Random Sampling Profit Extraction (RSPE) auction does. In particular, for a bid profile \(b = (b_1, \ldots, b_n)\), and unlimited items, the RSPE:

   (a) Assigns each bid uniformly at random into one of 2 sets, \(B_1, B_2\).

   (b) For \(k = 1, 2\), set \(R_k = \max_{i \in B_k} i \cdot b_k^{(i)}\), where \(b_k^{(i)}\) is the \(i\)th highest bid in \(B_k\).

   (c) (Given a bid profile \(b = (b_1, \ldots, b_n)\), with \(b_1 \geq \cdots \geq b_n\), and \(R > 0\) the profit extractor \(\text{Extract}_{R}(b)\) selects the maximum \(k\) for which \(b_k \geq R/k\). If such \(k\) exists, then each of the \(k\) highest bidders buy an item at price \(R/k\), and the total revenue is \(R\). Otherwise, no items are sold, and the revenue is 0.) The RSPE auction now runs \(\text{Extract}_{R_1}(B_2)\) and \(\text{Extract}_{R_2}(B_1)\).

   (d) The RSPE auction is strategyproof, and gives a 4-approximation for \(R_{\text{opt}}^{(2)}\) in a setting with an unlimited number of items (see Theorem 9.8).
3 Exercises

1. (a) Show that the uniform distribution on $[0,1]$ has a regular virtual value function by writing down the virtual value function. Do the same for the exponential distribution (remember that $G(x) = 1 - e^{-x}, x \geq 0$).

(b) Show that the distribution with c.d.f $G(x) = \sqrt{x}$, $0 \leq x \leq 1$, is not regular by comparing $\phi(0)$ and $\phi(1/9)$.

Answer:

(a) For uniform, the virtual value is $2v_i - 1$. For exponential, it is $v_i - e^{-v_i}/e^{v_i} = v_i - 1$.

(b) Here, $g(v_i) = 1/2 \cdot v_i^{-1/2}$, $0 \leq v_i \leq 1$, so $\phi(v_i) = v_i - 1 + \sqrt{v_i} = 3v_i - 2\sqrt{v_i}$. Note that $\phi(0) = 0$ but $\phi(1/9) = 1/3 - 2/3 = -1/3 < 0$, so $\phi$ is not monotone increasing.

2. Suppose that two bidders have values $v_1 \sim U(0,5), v_2 \sim U(0,10)$ (which are independent). Suppose they bid $b_1, b_2$.

(a) What are the virtual value function $\phi_1, \phi_2$?

(b) What are sufficient and necessary conditions for us to sell to bidder 1? If we sell, then that does bidder 1 pay (in terms of $b_2$)?

(c) What are sufficient and necessary conditions for us to sell to bidder 2? If we sell, then what does bidder 2 pay (in terms of $b_1$)?

Answer: (Note to TF: can draw this on board!)

(a) $\phi_1(v_1) = 2v_1 - 5, \phi_2(v_2) = 2v_2 - 10$.

(b) We must have $b_1 \geq b_2 - 5/2, b_1 \geq 5/2$. Bidder 1 pays $\max(5/2, b_2 - 5/2)$.

(c) We must have $b_2 \geq b_1 + 5/2, b_2 \geq 5/2$. Bidder 2 pays $\max(5, b_1 + 5/2)$.

3. For any integer $n$, give an example of a value profile $v$ with $n^2$ bidders, such that $R_{DOP}(v)/R_{opt}^{(2)}(v) = 1/n$, where $R_{DOP}$ is the revenue from the DOP auction. [Hint: some bidders should have value $n$, and the rest of the bidders should have value 1.]

Answer: Suppose there are $n^2$ bidders, $n$ with value $n$ and $n^2 - n$ with value 1. The revenue target is $R_{opt}^{(2)}(v) = n^2$, by selling $n$ items at a price of $n$ to each of the top $n$ bidders (or selling to all bidders, at a price of 1). The revenue from DOP is $n$, since each of the top $n$ bidders is charged 1, and each of the remaining bidders is charged $n - 1$, so does not buy. The ratio between these two quantities is $1/n$.

4. Consider the setting of an RSPE auction with 4 bidders, with bid profile $b = (b_1, b_2, b_3, b_4)$, with $b_1 = 6, b_2 = 3, b_3 = 4, b_4 = 1$. Suppose that in a particular random draw $B_1 = \{b_1, b_2\} = \{6, 3\}, B_2 = \{b_3, b_4\} = \{4, 1\}$.

(a) For the particular random draw above, find $R_1, R_2$.

(b) What is the outcome of the auction? In particular, run $\text{Extract}_{R_1}(B_2), \text{Extract}_{R_2}(B_1)$. Who is allocated what, and who pays what?

1If you’re bored, then try to classify the set of all distributions with linear virtual value functions.
(c) What do you notice about the allocation?

Answer:

(a) \( R_1 = 6, R_2 = 4 \).

(b) For \( Extract_{R_2}(B_1) \), we have \( b_1^{(1)} = 6, b_2^{(1)} = 3 \), and the maximum \( k \) for which \( b_k^{(1)} \geq R_2/k \) is \( k = 2 \). Then bidders 1, 2 each buy an item at price \( 4/2 \), and the total revenue from \( B_1 \) is 4.

For \( Extract_{R_1}(B_2) \), we have \( b_1^{(2)} = 4, b_2^{(2)} = 1 \), and there is no \( k \) for which \( b_k^{(2)} \geq R_1/k = 6/k \). So, bidders 3, 4 remain unallocated (and pay nothing).

(c) Bidder 3 is not allocated even though she has a higher bid than bidder 2! Moreover, only one of the two sets \( B_1, B_2 \) has bidders which are allocated.\(^2\)

\(^2\)Try to generalize this fact in some way.