1 Review

Key Points to Understand

1. VCG Mechanism: maximizes the total value of the agents and charges each agent the negative externality it imposes on the other agents by its presence. Formally, given reported valuation function \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \), the VCG mechanism is defined by

- A choice rule \( x(\hat{v}) \in \arg\max_{a \in A} \sum_{i \in N} \hat{v}_i(a) \)
- A payment rule s.t. \( t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(a^{-i}) - \sum_{j \neq i} \hat{v}_j(a^*) \) \( \forall \) agents \( i \)

where \( a^* = x(\hat{v}) \) and \( a^{-i} \in \arg\max_{a \in A^{-i}} \sum_{j \neq i} \hat{v}_j(a) \).
- The VCG mechanism is strategy-proof, efficient and satisfies participation.

2. Single Parameter Domain: each agent has a private input \( w_i \in \mathbb{R} \), and a summarization function \( q_i: A \rightarrow \mathbb{R} \) (known to the mechanism designer). For an allocation \( a \), agent \( i \)'s value is \( w_i \cdot q_i(a) \).

Examples include single item (possibly multi-unit) auction. A mechanism \((x, t)\) is strategy-proof for a single parameter domain if and only if, for all \( i \) and \( w_{-i} \)

- (monotonicity) \( q_i(x(\cdot, w_{-i})) \) is monotonically non-decreasing in \( w_i \)
- (payment identity) payment rule \( t \) satisfies

\[
t_i(w_i, w_{-i}) = w_i \cdot q_i(x(w_i, w_{-i})) - \int_{z=0}^{w_i} q_i(x(z, w_{-i})) \, dz
\]  

3. Several types of ads were discussed in class:

- **sponsored search**: advertisers bid on search-terms ahead of time (with standing bids). Each time a search term is entered, a position auction is run with eligible advertisers. Payments are usually made per-click. Generally GSP.
- **contextual ads**: generally keyword-based standing bids like sponsored search, only ads appear on external webpages that users visit, and are targeted based on the content of the webpage. Also behavioral targeting. Generally VCG.
- **display ads**: targeted based on page position and demographic (e.g., front page espn.com), increasingly sold through SPSB auctions in real-time ad exchanges.

4. In the **Generalized Second-Price (GSP)** auction, advertisers submit one bid each for multiple slots.
• Assumptions: Advertisers $i$ have uniform per-click values $v_i$, independent of position, number of clicks, etc. Each advertiser has an ad quality $Q_i$ (calculated by the auction designer). Each ad position $j$ has a position value $pos_j$ and each ad’s click-through rate is modeled by $CTR_{ij} = Q_i \cdot pos_j$. Effective values are given by $v_{ij} = CTR_{ij} \cdot v_i$.

• Allocation rule: Advertisers are ranked into slots according to decreasing $b_i \cdot Q_i$, where $b_i$ is the $i$th advertiser’s bid per click.

• Payment rule: Advertisers pay the least they would have needed to bid to keep their slots.

$$PPC_{gsp,i} = \frac{Q_{i+1}b_{i+1}}{Q_i}$$

where we assume that indices are ordered by effective bid order.

• The GSP auction is not strategy-proof. However, balanced bidding is a Nash equilibrium in GSP. Balanced bidding is defined by each advertiser bidding just high enough such that the next-highest advertiser bidding down to switch slots would not decrease the original advertiser’s utility. Bids in this equilibrium are value-ordered (generally by $Q_i \cdot v_i$). In the case of no quality scores, the BB equation is:

$$pos_{i-1}(v_i - b_i) = pos_i(v_i - b_{i+1})$$

for each advertiser $i \geq 2$ and $b_1 \geq b_2$.

5. The VCG position auction ranks the advertiser according to decreasing $Q_i \cdot b_i$. The expected payment by advertiser $i$ is given as

$$t_{vcg,i}(b) = \sum_{k=i+1}^{n} (pos_k - pos_{k+1})Q_kb_k$$

and the per click payment for advertiser $i$ is $t_{vcg,i}(b)/(Q_i \cdot pos_i)$. The VCG position auction is strategyproof and allocatively efficient. It is increasingly used for contextual ads.

In fact, the balanced bidding outcome in the GSP auction is the same as the truthful outcome in the VCG auction. Search engines still generally use GSP though (why?)

2 Practice

1. TV Advertising

Consider an auction where advertisers are bidding for Superbowl commercials. There are spots for three commercials to air and the iser with the first spot gets 40 views, the second gets 25 views, and the third gets 10 views (numbers in millions). Each advertiser wants one of the three commercials and only cares about the number of views she gets.

(a) Assume a mechanism designer runs an auction where bidders bid their willingness to pay per view. Come up with a choice rule that is monotone non-decreasing in every report $\hat{v}_i$.

(b) Consider the simple monotone non-decreasing choice and payment rules where the highest bidder receives the top spot and pays the second highest price per view; the second highest bidder receives the second spot and pays the third highest price per view, and the third highest
bidder receives the third spot and pays the fourth highest price per view. Is this mechanism strategy-proof? If so, show that it is strategy-proof, if not come up with an example where it is not strategy-proof.

(c) Come up with a payment mechanism that is strategy-proof using the payment identity from the reading on mechanism design (eq. 1). Suppose there are four advertisers participating in the auction and the bids of the first three agents are 4, 6, and 10. Draw the allocation vs value curve and the utility vs value curve for a fourth agent with valuation $0 \leq v_4 \leq 20$.

**Answer:**
(a) The highest bidder receives the top spot, the second highest receives the second spot, and the third highest receives the third spot. Break ties by awarding the higher spot to the agent with the lower index.

(b) No. This mechanism is not strategy-proof. Imagine a situation where four agents have valuations per view of 20, 19, 2, and 1. Under this mechanism, the player who bids 20 pays 19 on 40 views, giving him a total utility of $(20 - 19) \times 40 = 40$. However, if instead he were to bid 18, he would come in second, pay 2 and receive 25 views, giving him a total utility of $(20 - 2) \times 25 = 450$.

(c) First note that, given a valuation profile $v$ of the four agents, the allocation $x(v)$ is a four-dimensional vector where $x_i(v)$ is the slot allocated to ad $i$. For example, $x_4(v) = 1$ if ad 4 gets the first spot, 2 if she gets the second spot, 3 if she gets the third spot and 0 otherwise. Since the advertiser is only interested in number of views of an allocation, the summarization function maps the allocation to the corresponding number of views i.e.

$$q_4(x(v)) = \begin{cases} 
40 & \text{if } x_4(v) = 1 \\
25 & \text{if } x_4(v) = 2 \\
10 & \text{if } x_4(v) = 3 \\
0 & \text{if } x_4(v) = 0 
\end{cases}$$

Now, we want to use the payment identity. To start thinking about this, we draw a graph with the summarization function $q_4(x(v_4, v_{-4}))$ along the y-axis and $v_4$ of our fourth agent along the x-axis.
The payment identity gives us
\[
t_4(v_4, (4, 6, 10)) = v_4 \cdot q_4(x(z, v_{-4})) - \int_{z=0}^{z=v_4} q_4(x(z, v_{-4}))dz.
\]

For a given value of \(v_4\), the agent’s utility is \(v_4 \cdot q_4(x(v_4, v_{-4})) - t_4(v_4, (4, 6, 10))\) which is nothing but the area under the allocation curve from 0 to \(v_4\). The shaded area in the next graph shows the agent’s utility if his valuation is 12.

From the graph we can express utility of agent 4 as a function of \(v_4\).
\[
u_4(v_4, v_{-4}) = \begin{cases} 
0 & \text{if } 0 \leq v_4 \leq 4 \\
10(v_4 - 4) & \text{if } 4 < v_4 \leq 6 \\
20 + 25(v_4 - 6) & \text{if } 6 < v_4 \leq 10 \\
120 + 40(v_4 - 10) & \text{if } v_4 > 10 
\end{cases}
\]

The below graph illustrates his utility plotted against all values of \(v_4\).

(The same analysis in the case of the position auction leads to a well known auction design!)
2. GSP auctions

Consider the following position auction setting. Suppose our auction is a GSP auction with three positions and four bidding advertisers A, B, C, and D. The three positions have position effects $\text{pos}_1 = 0.3, \text{pos}_2 = 0.28, \text{pos}_3 = 0.1$. The bidders have per-click values (e.g. the customer value upon the click) $v_A = $100/click, $v_B = $50/click, $v_C = $18/click, and $v_D = $10/click.

(a) Suppose that the advertisers create ads with qualities $Q_A = 0.5, Q_B = 0.7, Q_C = 0.4, Q_D = 0.8$. If advertisers bid truthfully (i.e. bid $b_i = v_i$), what will the outcome be (i.e. what are the position allocation and payment for each advertiser)?

Answer: Allocations: ordering by position allocation and payment for each advertiser?

(b) From now on suppose that all advertisers have equal qualities $Q_i = 1$ for all $i \in \{A, \ldots, D\}$. Now what is the outcome of truthful bidding?

Answer: Allocations: $x_A = 1, x_B = 2, x_C = 3, x_D = 3$. Payments: each bidder pays $\frac{Q_i}{Q_i + \sum_{i=1}^{k} b_{i+1}}$, so $PPC_A = 70, PPC_B = \frac{80}{7}, PPC_C = 0, PPC_D = 9$.

(c) Is this truthful bidding a Nash equilibrium?

Answer: No, let advertiser A deviate and bid 30. Then advertiser A’s new expected utility $u_A' = p_2(v_A - b_3) = 0.28(100 - 18) > 0.3(100 - 50) = p_1(v_A - b_2) = u_A$.

(d) Suppose that we have bids $b_A = 20, b_B = 50, b_C = 18, b_D = 10$. Is this a Nash equilibrium? Is this outcome value-ordered (i.e. are the bids non-decreasing in the values)?

Answer: Yes NE. $u_A = 0.28(100 - 18) = 22.96, u_B = 0.3(50 - 20) = 9, u_C = 0.1(18 - 10) = 0.8, u_D = 0$. Check deviations for each advertiser. No, not value-ordered.

(e) The bids not being value-ordered in Nash equilibrium implies that this bidding outcomes does not satisfy the balanced-bidding condition. Check that the balanced-bidding condition is violated for some ad $i$.

Answer: In particular for ad $A$ allocated at slot 2, we need $\text{pos}_1(v_A - b_A) = \text{pos}_2(v_A - b_C)$, but $0.3(100 - 20) > 0.28(100 - 18)$. This also implies that, advertiser A could bid higher and still not risk retaliation from advertiser B that would hurt him.

(f) What is the balanced bidding outcome in this example? For this to be Nash, what is the key assumption we must make about the knowledge of the bidders?

Answer: $b_D = v_D = 10$. Now $b_C$ satisfies $\text{pos}_3(v_C - b_D) = \text{pos}_2(v_C - b_C) \Rightarrow b_C = \frac{106}{28}$. Similarly $\text{pos}_2(v_B - b_C) = \text{pos}_1(v_B - b_B) \Rightarrow b_B = \frac{262}{18}$. Pick any $b_A = v_A = 100$ and we’re done. The key assumption we make is that each bidder knows the others’ true values.

(g) Check that the balanced bidding outcome is Nash.

Answer: Yes. $u_A = 0.3(100 - \frac{262}{18}) = 0.28(100 - \frac{106}{7})$ and similarly any lower deviation. Also $u_B = 0.28(50 - \frac{106}{7}) > 0.1(50 - 10)$, and B would never try to outbid A. The check for C is similar.

3. VCG in Online Advertising

Consider the following position auction setting. Suppose our auction is a VCG auction with three positions and four bidding advertisers A, B, C, and D. The three positions have position effects $\text{pos}_1 = 0.3, \text{pos}_2 = 0.28, \text{pos}_3 = 0.1$. The bidders have per-click values (e.g. the customer value upon the click) $v_A = $100/click, $v_B = $50/click, $v_C = $18/click, and $v_D = $10/click.

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Answer: No, let advertiser A deviate and bid 30. Then advertiser A’s new expected utility $u_A' = p_2(v_A - b_3) = 0.28(100 - 18) > 0.3(100 - 50) = p_1(v_A - b_2) = u_A$.

(d) Suppose that we have bids $b_A = 20, b_B = 50, b_C = 18, b_D = 10$. Is this a Nash equilibrium? Is this outcome value-ordered (i.e. are the bids non-decreasing in the values)?

Answer: Yes NE. $u_A = 0.28(100 - 18) = 22.96, u_B = 0.3(50 - 20) = 9, u_C = 0.1(18 - 10) = 0.8, u_D = 0$. Check deviations for each advertiser. No, not value-ordered.

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effects \( pos_1 = 0.3, pos_2 = 0.28, pos_3 = 0.1 \). The bidders have per-click values (e.g. the customer value upon the click) \( v_A = \$100/\text{click}, v_B = \$50/\text{click}, v_C = \$18/\text{click}, \) and \( v_D = \$10/\text{click} \).

(a) Suppose that the advertisers create ads with qualities \( Q_A = 0.5, Q_B = 0.7, Q_C = 0.4, Q_D = 0.8 \). What are the position allocation and payment for each advertiser if advertisers bid truthfully?

Answer: Allocations: \( x_A = 1, x_B = 2, x_C = \text{unallocated}, x_D = 3 \). The payment of each bidder is \( \frac{1}{Q_i \cdot pos_i} \left( \sum_{k=i+1}^{m+1} (pos_{k-1} - pos_k) Q_k b_k \right) \). Payments: \( PPC_A = 19.067, PPC_B = 11.020, PPC_C = 0, PPC_D = 9 \).

(b) From now on suppose that all advertisers have equal qualities \( Q_i \) for all \( i \in \{A, B, C, D\} \). Now what is the outcome of truthful bidding?

Answer: Allocations: \( x_A = 1, x_B = 2, x_C = 3, x_D = \text{unallocated} \). The payment of each bidder is \( \frac{1}{Q_i \cdot pos_i} \left( \sum_{k=i+1}^{m+1} (pos_{k-1} - pos_k) Q_k b_k \right) \). Payments: \( PPC_A = 17.467, PPC_B = 15.143, PPC_C = 10, PPC_D = 0 \).

(c) Is this VCG auction strategy-proof? Is there any useful deviation for advertisers? Check it for advertiser A in this example.

Answer: Yes. If advertiser A bids higher than \$50/\text{click}, the payment doesn’t change and the utility is the same. If he bids higher than \$18/\text{click} and lower than \$50/\text{click}, since \( 0.3(100 - 17.467) > 0.28(100 - 15.143) \), thus it is not a useful deviation. Similarly, we can prove the deviation between \$10/\text{click} and \$18/\text{click} is also not useful.