1 Important Points

1.1 Key Ideas

1. A Combinatorial Auction (CA) is an auction where bidders \( N = \{1, \ldots, n\} \) bid on bundles of distinct, indivisible goods \( G = \{A, B, C, \ldots\} \), \(|G| = m\). An efficient CA allocates items to maximize the sum of the value of bidders. An allocation \( X = (X_1, \ldots, X_n) \) allocates \( X_i \subset G \) to bidder \( i \in N \), and is feasible if each item is allocated to at most one bidder. An efficient allocation solves

\[
\max_{X} \sum_{i=1}^{n} v_i(X_i) \\
\text{s.t. } X_i \cap X_j = \emptyset, \text{ for all } i, j \in N, i \neq j
\]

2. Combinatorial Auctions are especially useful in cases where valuation functions for packages are not simply additive in the value of the items. Two examples of such valuation functions include subadditive and superadditive valuation functions. A superadditive function exhibits complements, and satisfies

\[
v_i(S \cup S') \geq v_i(S) + v_i(S')
\]

for all disjoint packages \( S \) and \( S' \), and a subadditive function exhibits substitutes, with

\[
v_i(S \cup S') \leq v_i(S) + v_i(S')
\]

for all disjoint packages \( S \) and \( S' \).

3. Some examples of actual CAs include allocating airport landing spots and allocating spectrum frequencies.

Imagine if airport landing spots were each auctioned off in sequence as single good auctions. An airline carrier might win gates A2, A5, B4, C7 (in different terminals) which is inefficient for business. The airline would much rather win a connected block of gates all in the same terminal. A combinatorial auction allows airlines to express their desires for contiguous gates when bidding.

4. There are several bidding languages in CAs: OR, XOR, OR*, OR-of-XOR, XOR-of-OR. Some desirable properties of bidding languages include being expressive and succinct. The OR* language is fully expressive and exponentially succinct.

5. Solving the optimal allocation problem in a CA is much harder than for the single good auctions and can be formulated as an Integer Program (IP). WEIGHTED-INDEPENDENT-SET can be reduced to winner allocation and is NP-hard.
6. The Winner Determination Problem is tractable if all of the bids are the same type out of the following structural properties: cyclic structure, tree structure, pair bids, or hierarchical structure. Another way to make the winner determination problem tractable is to modify the bidding language.

7. VCG can be used to allocate and price bundles in CAs, but there are lots of problems: low revenue, vulnerability to loser collusion, false-name problems, etc.

2 Problems

2.1 CAs

1. Bidding Languages:

I am willing to pay $4 for item X, $3 for item Y, and in addition, I am willing to pay $2 for Z if and only if I win X but not Y. If possible, express these preferences using OR, XOR, and OR with a dummy item.

2. (Extra Interest) Consider a VCG mechanism applied to a combinatorial auction with three goods \{A, B, C\} and bids \((A, 10), (B, 4), (C, 6), (AB, 15), (BC, 12), (ABC, 19)\), all by different bidders.

(a) What is the outcome of the VCG mechanism in this example? Who is allocated the goods and what price(s) are paid?

(b) Consider running a greedy algorithm that selects compatible bids in order based on the bid price divided by the square root of the number of items in the bid. What is the outcome of the greedy algorithm? How efficient is this algorithm?

(c) Show that if we used VCG-style payments, i.e. charged each winner the net increase in utility of the other bidders if the greedy algorithm had been run without her, this auction would not be truthful. Then determine what the critical value payment would be.

3. Consider a weighted undirected graph \(G = (V, E)\) shown in Figure 1, where the number in each vertex is the weight of it. Then show how to construct OR bids in the corresponding CA and write down the integer programming.

![Weighted undirected graph and the corresponding CA.](image)