CS 136 Assignment 3: Algorithmic Game Theory

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Due 5pm sharp: Friday Feb 19, 2016
Submissions to Canvas

Total points: 40. This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of answers, and we encourage typed submissions. You are free to discuss the problem set with other students but you must not share your answers. Extra credit will only be considered as a factor in deciding the letter grade for the course at the end of the term.

1. [17 Points] Nash is a problem in PPAD.
   
   (a) [3 Points] Where not shown in Example 3.5, verify the labels $\ell_1$ and $\ell_2$ in Figure 3.9 are valid. Also give three additional $\ell_1$s that are NOT valid.
   
   (b) [3 Points] Draw the PPAD graph for the game in Figure 3.6 for base action $a_b = D$.
   
   (c) [1 Points] Why might following the path from the standard source not be a useful algorithm for finding a Nash equilibrium? (1-2 sentences is fine)
   
   (d) [4 Points] Explain why it is tractable to check whether a label corresponds to a vertex.
       [Hint: assume the game is non-degenerate, and give an LFP for each player. In a non-degenerate game, player 2 has at most $k$ best-response actions to any mixed strategy of player 1 with support of size $k$ (and vice versa).]
   
   (e) [2 Points] The set of correlated equilibria include all Nash equilibria. Why does the existence of polynomial-time algorithms for finding a correlated equilibrium not show that PPAD = P? (1-2 sentences is fine.)
   
   (f) [4 Points] Use the fact that (in a non-degenerate game) any Nash equilibrium must correspond to a completely-labeled vertex in the PPAD graph to argue that the size of the supports of the mixed strategies of each player must be the same in any Nash equilibrium (e.g., the support is size two for both players in the (2/3,1/3,0),(1/3,2/3) Nash equilibrium in the game in Example 3.5.)

(*) (extra credit) Prove that every vertex with an almost complete label has exactly two neighbors and every vertex with a complete label has exactly one neighbor (assume a non-degenerate game).

2. [13 Points] Game gadgets for performing calculations.
   
   (a) [3 Points] MULTIPLY: Figure 3.10 provides a 4-agent game gadget for multiplication. Why does a simpler 3-agent gadget, with $u_3(a_3) = a_1a_2$ if $a_3 = 1$ and $u_3(a_3) = 1 - a_1a_2$ if $a_3 = 0$, fail?
(b) [2 Points] THRESHOLD: Design a 2-agent gadget, where \( x_1, x_2 \in [0, 1] \) is the mixed strategy of agents 1 and 2 respectively, to implement \( x_2 = 1 \) if \( x_1 > 1/2 \), and \( x_2 = 0 \) if \( x_1 < 1/2 \). The behavior can be anything for \( x_1 = 1/2 \). Agent 1 has zero payoff whatever agent 2 does, and represents the input.

(c) [3 Points] Design a 3-agent gadget, where \( x_1, x_3 \in [0, 1] \) are the mixed strategies of agents 1 and 3, respectively, to implement \( x_3 = \min(\alpha x_1, 1) \) for some constant, \( \alpha \geq 0 \). The output agent 3's payoff structure should be 1 when it takes the opposite action to 2. Agent 1 has zero payoff for all actions of the other agents, and represents the input. Design the payoff for the middle agent 2, and prove that the required output property holds in every Nash equilibrium. [Hint: the analysis should follow the structure of that for the multiplication gadget, and consider cases \( x_3 > \alpha x_1 \) and \( x_3 < \min(\alpha x_1, 1) \).]

(d) [3 Points] Design a 4-agent gadget, where \( x_1, x_2 \) and \( x_4 \in [0, 1] \) are the mixed strategies of agents 1, 2 and 4, respectively, to implement \( x_4 = \min(x_1 + x_2, 1) \). Agent 4 represents the output, with payoff 1 when it takes the opposite action to agent 3, and 0 otherwise. Agents 1 and 2 represent the inputs, and have zero payoff for all actions of the other agents. Design the payoff for the middle agent 3, and prove the required output property holds in every Nash equilibrium. [Hint: follow the same kind of approach as in part (c).]

(e) [2 Points] Design a 2-agent gadget, to implement \( x_2 = \alpha \) for some constraint, \( \alpha \in [0, 1] \). The output agent 2's payoff structure should be 1 when it takes the opposite action to agent 1. Design the payoff for agent 1, as a function of the action of agent 2, and prove the required output property holds in every Nash equilibrium.

3. [10 Points] Zero-sum games

(a) [2 Points] Adopt the graphical approach in Figure 3.2 to identify the maximin strategy for player 1 in the Matching Pennies game in Chapter 2.

(b) [4 Points] Construct an example of a zero-sum two-player game where one or more players has multiple maximin strategies, and show that all combinations of strategies correspond to Nash equilibria.

(c) [3 Points] Prove that any two-player constant-sum game, with \( u_1(a) + u_2(a) = c \) for some constant \( c \in \mathbb{R} \), for all action profiles \( a \in A \), is equivalent to a two-player zero-sum game, in the sense that it has the same set of mixed-strategy Nash equilibria. [Hint: the set of mixed-strategy Nash equilibria in a game are invariant to positive affine transformations of payoffs.]

(d) [1 Points] What does this analysis show about the complexity of NASH in constant-sum games?

4. [Extra credit] Potential games

(a) Show that the game of Chicken is a potential game, and construct a slight variation that illustrates that asymmetric games with 2 players and 2 actions can be potential games.

(b) Construct a 2 player, 2 action game that has the finite-improvement property but is not a potential game. [Hint: it will need to have a pure-strategy Nash equilibrium in which an agent is indifferent between playing the equilibrium and deviating.]
(c) Construct a 2 player game with a dominant-strategy equilibrium that is not a potential game.

(d) Consider a potential game $G$, and a game $G'$ in which every player's utility is $u_i(a) = Pot(a)$, and equal to the potential function in game $G$. Do games $G$ and $G'$ have the same set of Nash equilibria? Why or why not?

(e) Prove that if all cycles in a potential game have zero value then the game is a potential game. [Hint: fix any action profile $z$, and as a first step, establish by the zero-value-cycle property that any two paths from $z$ to an action profile $a \neq z$ have the same value. Second, show that $Pot(a) = I(z \to a)$, where $I(z \to a)$ is the value of any path from $z$ to $a$, satisfies potential property (3.24).]

(f) Prove that if all 2-by-2 cycles have zero value then all cycles have zero value. [Hint: assume for contradiction that there is a cycle $\gamma = (a^{(0)}, a^{(1)}, \ldots, a^{(\ell-1)}, a^{(\ell)})$, where $a^{(\ell)} = a^{(0)}$, of length $\ell \geq 5$, with value $I(\gamma) \neq 0$, and that this positive-value cycle is minimal, in that all cycles of length $\ell - 1$ have zero value. Assume WLOG that agent 1 moves in step 0, and let $j$ denote another step in which 1 must move (this is required for it to be a cycle). First, argue by minimality (or the 2-by-2 assumption if $\ell = 5$) that $j$ is not step 1 or $\ell - 1$. WLOG, suppose agent 2 moves in step $j - 1$. Now consider cycle $\gamma'$, which differs from $\gamma$ only in that agent 1 now deviates in step $j - 1$ and agent 2 in step $j$; i.e., steps $a^{(j-1)}, a^{(j)}, a^{(j+1)}$ in $\gamma$ become steps $a^{(j-1)}, z^{(j)}, a^{(j+1)}$ in $\gamma'$, where $z^{(j)}$ is obtained from $a^{(j-1)}$ by agent 1’s deviation. Second, argue by the zero-value-2-by-2 property that $I(\gamma) = I(\gamma')$. Third, by considering minimality, and recognizing $I(\gamma') \neq 0$, complete the proof.]