1 Sequential-Move Games

1.1 Review

In sequential-move games, a player’s strategy needs to describe the action he will take after every sequence of observations about the actions of other players:

1. **Extensive form of sequential-move game:** The extensive form is defined by
   
   (a) A set of players \( N = \{1, \ldots, n\} \)
   
   (b) A set \( H \) of action histories \( h \) (with empty history \( \varepsilon \))
   
   (c) A set \( Z \subset H \) of terminal histories and a utility \( u_i(h) \) for each player \( i \) and \( h \in Z \)
   
   (d) A mapping from each non-terminal history \( h \in H \setminus Z \) to a player \( i = P(h) \in N \) whose turn it is to move, and action set \( A_i(h) \) available to the agent.

2. **Strategies in extensive form:** The strategy \( s_i \) in an extensive form game defines \( s_i(h) \in A_i(h) \) for every non-terminal history \( h \in H \setminus Z \) for which it is agent \( i \)'s turn to play. (Analogy: a player writes down his contingency plan for every situation before the game starts.)

3. **Subgame-perfect Equilibrium:** A strategy profile \( s^* = (s_1^*, \ldots, s_n^*) \) is a subgame-perfect equilibrium if it is a Nash equilibrium in the subgame at every non-terminal history \( h \in H \setminus Z \).

1.2 Practice

Consider the extensive form game as shown in Figure 1 (see next page).

1. Write down the set \( H \) of action histories \( h \) of this game. Which elements of \( H \) belong to the set \( Z \) of terminal histories?

2. Find \( P(\varepsilon), A_1(\varepsilon) \) and \( P((Y)) \).
3. Find the pure strategy Nash equilibria of the game by writing out the normal form representation of the game.

4. Use backward induction to find the pure strategy subgame-perfect equilibrium.

5. Give an example of a Nash equilibrium that is not subgame-perfect in this game.

2 Single-deviation Principle

Single-deviation Principle A strategy profile is a subgame-perfect equilibrium in a finite extensive form game iff there is no subgame at non-terminal history $h$ where the agent whose turn it is to move has a useful deviation by changing his action at that history $h$ only.

2.1 Practice

Consider the repeated entrance deterrence game. Player 1 decides whether to enter a market, then player 2 decide whether to accommodate or fight agent 1.  

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1. Consider the stage game as shown in Figure 2. Find the Nash equilibria of the game. Which one(s) of them is/are SPE?

![Figure 2: The Entrance Deterrence Game](image)

2. Suppose the stage game is repeated twice, and the payoffs from the two rounds are added directly, as shown in Figure 3. Use backward induction to find a SPE of the game.

![Figure 3: The Repeated Entrance Deterrence Game](image)

3. Describe player 2’s strategy in extensive form under this SPE. How about player 1?

4. What does a single deviation of player 2 at \( h = \text{(enter)} \) mean? What happens after this single deviation, and what are the agents’ payoffs?

5. Describe some deviation from the SPE of player 2 that is not “single”, and briefly explain why single deviation principle works.
3 Repeated Games

3.1 Review

In a repeated game $G^T$, the same simultaneous move stage game $G = (N, A, u)$ is played by the same players for $T$ periods, with every agent having perfect information about the history of actions. In an infinitely repeated game $G^\infty$ the stage game $G$ is repeated forever:

1. **Utility and discounting:** A player's total utility is the sum of his utilities across stage games. In this sum, a player's utility from game $k$ may be discounted by $\delta^k$.

2. **Unique subgame-perfect equilibrium:** If the stage game has a unique Nash equilibrium, then a finitely repeated game has a unique subgame-perfect equilibrium, which is to play the stage game Nash equilibrium in every period.

3. **Automation strategy:** An automaton strategy $m_i$ for player $i$ is defined by
   
   (a) A set $Q_i$ of machine states
   (b) A start state $q_0^i \in Q_i$
   (c) A next state function $q_i' = \text{succ}_i(q_i, a)$ for all states $q_i$ and action profiles $a \in A$
   (d) A mapping from states to actions $f_i : Q_i \rightarrow A_i$.

3.2 Practice

Consider the *tit for tat* (TfT) automation strategy in an infinitely repeated Prisoner’s Dilemma, in which a player would first cooperate, then subsequently replicate an opponent’s previous action. Assume the payoff matrix is as follows:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(3, 3)</td>
<td>(0, 5)</td>
</tr>
<tr>
<td>D</td>
<td>(5, 0)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

1. Draw the finite-state automaton for the TfT strategy.

2. Why is (TfT, TfT) not subgame perfect?

3. Is (TfT, TfT) a Nash equilibrium of this game, and for which discount factors $\delta \in [0, 1)$?
4 Open-Loop Strategy

4.1 Review

**Open-loop strategy:** An open-loop strategy $s_i$ for player $i$ in a repeated game satisfies $s_i(h) = s_i(h')$ for all histories $h, h'$ of the same length. In particular, a player’s action can depend on the number of periods since the start of the game but not on the actions in previous periods.

**Example** Open-loop strategies in the repeated game of Chicken

<table>
<thead>
<tr>
<th>Player 1 \ Player 2</th>
<th>Y</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>(0, 0)</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>S</td>
<td>(2, 0)</td>
<td>(-4, -4)</td>
</tr>
</tbody>
</table>

- go straight in every period
- yield in every period
- cycle through straight, straight, yield every three periods.

**Theorem** (Thm 4.5 in the readings) An open-loop strategy profile in which a Nash equilibrium of the stage game is played in each period of a finite or infinitely repeated game is a subgame-perfect equilibrium of the repeated game.

4.2 Practice

1. Prove the above theorem.

2. Show that agent 1 cycles through straight, straight, yield and agent 2 cycles through yield, yield, straight is a subgame-perfect equilibrium of the repeated game of Chicken.

5 Folk Theorem

5.1 Review

**Enforceable action** An action profile $a$ of the stage game is enforceable if there exists a (possibly mixed) Nash equilibrium strategy $s^*$ of the stage game, for which

$$\tilde{u}_i(a) > \tilde{u}_i(s^*)$$

for all agents $i$. 
Folk Theorem  Given a stage game $G$ with an enforceable action profile $a$, there exists a subgame perfect equilibrium of the infinitely repeated game $G^\infty$, for all sufficiently large $\delta$, where action profile $a$ is played in equilibrium in every period.

- **Cooperate**: Play according to $a$ if everyone played according to $a$ in all previous periods.
- **Punish**: Otherwise, play strategy $s^\ast$ in every subsequent round.

5.2 Practice

1. Is there any enforceable action profile in the game of chicken?

2. 3 Player Pollution Game. Assume there are 3 players and a river that is polluted. Cleaning the river costs 1 for each player and the benefit from a clean river is 3 for each player. However, at least two players are needed to clean the river. Denote “cleaning” as C and “do not clean” as D.

   1) Is (D, D, D) a NE of the game?

   2) Is (C, C, C) a NE of the game? Why or why not? Is it an enforceable action profile?

   3) Describe an SPE that enforces (C, C, C) and find the minimum $\delta$ required for this to hold.

4) Is (C, C, D) an NE? Is it enforceable? Is playing (C, C, D) at every stage an SPE? Describe two SPE’s s.t. (C, C, D) is “played in every period of the equilibrium”.

6
5) Some extra exercises relating to the previous section.

(1) What’s the size of the normal-form representation of the game? Write out the payoff table and find all pure strategy Nash equilibria. (Hint: there are a total 4 of them).

(2) Is this a congestion game?

6  P2P File Sharing

6.1 Review - History

BitTorrent: Torrent files contain metadata for a file, and the IP of a tracker. That tracker holds information about the swarm, the group of users currently downloading/uploading that file. Users can request a list of roughly 50 peers from the tracker, to begin downloading the file.

Peers exchange information with one another about which pieces of the file they each have. Each person will download from any/all peers that will upload to them. But, they will only upload to those peers that meet certain conditions.

In the reference client, there are four upload slots. The first 3 slots are given to the peers from whom you have the highest download rate. The 4th is for ‘optimistic unchoking’, which is given to a random peer with the hope that they will reciprocate.

Exploits: under-reporting pieces available for upload, strategic unchoking, uploading garbage (effectively prevented by hashing blocks), growing neighborhood quickly and free-riding (effectively prevented by having trackers limit multiple requests from the same IP).

6.2 Practice

1. Explain why breaking files into small pieces helps the overall performance of the system?

2. Suppose some reference client for BitTorrent was near-universally adopted, and was a NE against itself. Why might you still worry that this is insufficient to ensure stability of the ecosystem?