• Gibbard Satterthwaite impossibility
  – 3 or more outcomes
  – All strict preference orders possible
  – onto and truthful => dictatorial
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• “truthful mechanisms are uninteresting”

• What to do?
  – Restrict preference domain
  – Two outcomes
  – Not truthful
• Gibbard Satterthwaite impossibility
  – 3 or more outcomes
  – All strict preference orders possible
  – onto and truthful => dictatorial
• “truthful mechanisms are uninteresting”

• What to do?
  – Restrict preference domain
  – Two outcomes
  – Not truthful

Restricted Preference Domains

• Two-sided matching
  – Outcome?
  – All strict preferences possible?
• House allocation
  – Outcome?
  – All strict preferences possible?
• Paired-Kidney Donation
  – Outcome?
  – All strict preferences possible?
I: Two-sided Matching

- Students to (public) schools; Medical interns to residencies; TAs to professors

- Agents in two sets; agent in one set has strict preferences over agents in other set

- A matching: each agent assigned to at most one agent on other side

Example
An unstable matching

the boys and girls

Jake ♥ Holly > Claire > Twiggy > Jill

Elwood Claire > Jill > Twiggy > Holly

Curtis Twiggy > Jill > Holly > Claire

Ray Holly > Claire > Twiggy > Jill

Twiggy Jake > Elwood > Curtis > Ray

Claire Jake > Curtis > Elwood > Ray

Jill Ray > Curtis > Elwood > Jake

Holly Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)

Stable matching

• A matching with no **blocking pair**
• A **blocking pair** = two agents who prefer each other to match
Some questions

• Do stable matchings exist?
• Are they easy to find?
• Does stability matter?
• Are stable matchings unique?
• What about incentives?

Boy-Proposing Deferred Acceptance (Gale-Shapley 1962)

Round 1

Jake
Holly > Claire > Twiggy > Jill

Elwood
Claire > Jill > Twiggy > Holly

Curtis
Twiggy > Jill > Holly > Claire

Ray
Holly > Claire > Twiggy > Jill

Round 2

Twiggy
Jake > Elwood > Curtis > Ray

Claire
Jake > Curtis > Elwood > Ray

Jill
Ray > Curtis > Elwood > Jake

Holly
Ray > Jake > Elwood > Curtis

Round 3

Stop! (Stable)

(Immorlica/Hartline)
Stable matchings exist

• Thm: Boy-proposing DA generates **stable matching**

*Proof.*

• Suppose \((b,g), (b',g')\) in the matching
• For contradiction, suppose \((b,g')\) blocking pair
• \(\Rightarrow\) boy \(b\) prefers \(g'\) to \(g\), and proposes to \(g'\) before girl \(g\)
• \(\Rightarrow\) girl \(g'\) prefers boy \(b'\) to boy \(b\) (since match improves for girls during the DA)
• \(\Rightarrow\) \((b,g')\) is **not** a blocking pair. Contradiction.

---

Some questions

• Do stable matches exist? **yes**
• Are they easy to find? **yes**
• Does stability matter?
• Are stable matches unique?
• What about incentives?
Stability looks like an important feature of a centralized labor market clearinghouse.

Some questions

- Do stable matches exist? **yes**
- Are they easy to find? **yes**
- Does stability matter? **yes**
- Are stable matches unique?
- What about incentives?
Girl-Proposing Deferred Acceptance

Round 1  Round 2  Stop! (Stable)

Jake  Holly > Claire > Twiggy > Jill
Elwood  Claire > Jill > Twiggy > Holly
Curtis  Twiggy > Jill > Holly > Claire
Ray  Holly > Claire > Twiggy > Jill

Jake > Elwood > Curtis > Ray
Jake > Curtis > Elwood > Ray
Ray > Curtis > Elwood > Jake
Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)

Not unique!

Jake  Holly > Claire > Twiggy > Jill
Elwood  Claire > Jill > Twiggy > Holly
Curtis  Twiggy > Jill > Holly > Claire
Ray  Holly > Claire > Twiggy > Jill

Jake > Elwood > Curtis > Ray
Jake > Curtis > Elwood > Ray
Ray > Curtis > Elwood > Jake
Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)
Some questions

- Do stable matches exist? yes
- Are they easy to find? Yes
- Does stability matter? Yes
- Are they unique? No
- What about incentives?

Achievable outcomes

In general, there are many stable matchings.

A girl $g$ is achievable for $b$ if $b$ and $g$ match in some stable matching

e.g., $\{g, g\}$ are achievable for $b$ (Immorlica/Hartline)
In boy-proposing DA:
1. Boys match with their most preferred, achievable girls
2. Girls match with their least preferred, achievable boys

*vice versa* for girl-proposing DA.

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Incentives

- Is truthful reporting a dominant strategy for boys in boy-proposing DA?
Incentives

• Is truthful reporting a dominant strategy for boys in boy-proposing DA?
• Yes. If truthful, boy $b$ is matched to most-preferred, achievable girl; cannot do better.

• Girl-proposing DA is truthful for girls.
Incentives

• Is truthful reporting a dominant strategy for boys in boy-proposing DA?
  • Yes. If truthful, boy $b$ is matched to most-preferred, achievable girl; cannot do better.

• Girl-proposing DA is truthful for girls

• Negative result: No matching mechanism is stable and truthful (for both sides of the market).

Some questions

• Do stable matches exist? yes
• Are they easy to find? Yes
• Do stable matches matter? Yes
• Are they unique? No
• What about incentives? Yes, on one side.
Real-world Matching Markets

**National Residency Matching Program (NRMP).**

- Adoption of student-proposing NRMP in 1998
- Easier for students
- Practical concern: couples with preferences on pairs of positions

**School Choice (Boston and New York).**

- "Boston mechanism" was not stable or truthful
- Fix: adopt student-proposing DA
- Easier, more fair, and allow for policy advice
- Practical concern: priorities for schools (siblings, walk zones)
II. One-sided Matching

• “House allocation” problem
  – No initial allocation

• “Housing market” problem
  – Initial allocation of items

• Agents have strict preferences on items
• Items don’t have preferences on agents 😊
House allocation

- Set of items
- Each agent has strict preferences on items
- Want truthful, Pareto optimal mechanism
- Solution?
- How make this fair?

House allocation

- Set of items
- Each agent has strict preferences on items
- Want truthful, Pareto optimal mechanism
- Solution? *Serial dictatorship (SD)*
- How make this fair? *Random SD*
House allocation

- Set of items
- Each agent has strict preferences on items
- Want truthful, Pareto optimal mechanism
- Solution? Serial dictatorship (SD)
- How make this fair? Random SD
  Conjectured to be only anonymous, Pareto optimal, truthful mechanism.

Housing Market Problem

Each agent has a house.

Strict preference order on houses.

(Hartline/Immorlica)
Top-trading cycle mechanism

1. Each agent “points” to most preferred house (allow self-edge).
2. Trade on cycles, agents and houses leave market.
3. Each remaining agent points to its most preferred, remaining house.
4. Repeat (#2, #3), until no agents left.

(Hartline/Immorlica)

TTC mechanism

(Hartline/Immorlica)
TTC mechanism

House A > House B > House D > House C

(TTC mechanism)
TTC mechanism

What if there are no cycles? What if an agent is on multiple cycles?

Top trading cycles

Each agent has one out-edge → there is at least one cycle, cycles don’t “conflict.”

(Hartline/Immorlica)
Top trading cycles

1. Each agent points to her most preferred house that is still on the market.

2. Select a cycle and trade, agents that trade leave the market.

(Hartline/Immorlica)

Truthful
Pareto optimal
Core: no subset prefer to trade amongst themselves

(Hartline/Immorlica)
III: Kidney-Paired Donation

Kidney disease

• **Kidney failure** serious medical problem
• Preferred treatment: **kidney transplant**
  – Cadaver kidneys
  – Donation from live patient
  – Must be blood- and tissue-type compatible
At this time:

121,274 people
waiting for a kidney transplant in the US.

http://optn.transplant.hrsa.gov

In 2008,

10,526 patients
received cadaver kidneys.

4,857 patients
received live donor kidneys.

5,920 patients died
or became too sick for a transplant.

(Hartline/Immortica)
Legality

Section 301 of the National Organ Transplant Act, “Prohibition of organ purchases” imposes criminal penalties on any person who

“knowingly acquire[s], receive[s], or otherwise transfer[s] any human organ for valuable consideration for use in human transplantation”

Kidney-Paired Donation

Incompatible pairs arrive into market.
- a donor and a patient
- participate in swaps, cycles…

Trade “like for like.” Legal.
Swap (2-cycle)

1. Pairs register in database.
2. Form a graph, representing possible compatibilities
3. Feasible matchings found
4. Additional medical tests (“cross match”)
5. Transplants performed

Kidney exchanges (APD, UNOS, NKR)
Match Offer Process

Modeling KX as a Housing market?

- Each pair “points” to its most preferred other pair
- Find cycles, “trade.” Repeat.
Modeling KX as a Housing market?

• Each pair “points” to its most preferred other pair
• Find cycles, “trade.” Repeat.

• Doesn’t work:
  – 0/1 preferences in KX, not strict preferences
  – Limits on cycle lengths. Why?

Kidney-Paired Donation

• Find vertex-disjoint cycles of length <= k that cover as many vertices as possible

(Conitzer)
Special case: \( k=2 \)

- If edges go in both directions, replace by undirected edge
- Remove other edges

Find \textit{max cardinality matching} (max \#edges, every vertex incident on at most one edge)
- Edmond's algorithm (poly time.)

**Complexity**

- \( k = 2 \): in \( \mathbb{P} \) by Edmonds alg
- \( k = \infty \) in \( \mathbb{P} \)
  
  (via a reduction to a maximum weight perfect matching problem)
Complexity

• $k = 2$: in $\mathbf{P}$ by Edmonds
• $k = \infty$ in $\mathbf{P}$
• But $k = 3, 4, 5, \ldots$: $\mathbf{NP}$-hard!

Cycle formulation (IP)
Altruistic Non-simultaneous Donor Chains

• “Good Samaritan” donors. Enable (long) chains
Altruistic Non-simultaneous Donor Chains

• “Good Samaritan” donors. Enable (long) chains

![Graph diagram showing a kidney transplant chain with donors and recipients]

• Computational problem challenging
  – can reduce “cycles and chains” problem to “cycles problem” by including zero-weight “back edges” to D*

National Kidney Register

Completed 500+ transplants
Entirely “altruistic non-simultaneous chains”

• 60 person, 30 transplant chain (8/11- 12/11)
