Example Midterm Questions I
1. Game theory I

Consider the repeated *Stag hunt* game:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S</strong></td>
<td>4, 4</td>
<td>0, 1</td>
</tr>
<tr>
<td><strong>H</strong></td>
<td>1, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Player 1

(a) **[3 Points]** Give an example of a subgame-perfect Nash equilibrium strategy in an infinitely repeated Stag hunt game in which the equilibrium play includes both *S* and *H*. Briefly justify your answer.

(b) **[3 Points]** Modify the payoff for one action profile in the stag hunt game so that, when repeated a finite number of times, the game has a unique subgame-perfect Nash equilibrium. Briefly justify your answer.
2. **Game theory II**

Consider the following game.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>(3,4)</td>
</tr>
<tr>
<td>Center</td>
<td>(4,6)</td>
</tr>
<tr>
<td>Down</td>
<td>(2,3)</td>
</tr>
</tbody>
</table>

(a) **[2 Points]** Which strategies can be eliminated through iterated eliminated of strictly dominated strategies?

(b) **[2 Points]** Using only the remaining strategies, solve for the pure strategy Nash equilibrium/equilibria or argue that none exist.

(c) **[2 Points]** Solve for the mixed strategy Nash equilibrium.
(d) [2 Points] Suppose now the game is played as a sequential game. If you are player 1, would you prefer to move first or second, or are you indifferent?
3. Algorithmic Game Theory

(a) [2 Points] Pure-strategy Nash equilibrium can be computed in polynomial time in the size of the normal-form representation of a game. This statement is true, but its implication for algorithmic game theory is limited for two reasons. What are they?

(b) [2 Points] Consider a joint probability distribution $p^*$ on action profiles $A$, with $p^*(a_{-i} | a_i)$ to denote the conditional probability of action profile $a_{-i}$ by others given action $a_i$ by agent $i$, and $p^*(a_i) = \sum_{a_{-i} \in A_{-i}} p^*(a_i, a_{-i})$ to denote the marginal probability with which agent $i$ plays action $a_i$.

There is one mistake in the following statement. What is it?

*Joint probability distribution $p^*$ is a correlated equilibrium of a game if, for all agents $i$ and all actions $a_i \in A_i$ with $p^*(a_i) > 0$:

$$\sum_{a_{-i} \in A_{-i}} p^*(a_{-i} | a_i) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} p^*(a_{-i} | a'_i) u_i(a'_i, a_{-i}), \quad (1)$$

for all actions $a'_i \in A_i$."

(c) [3 Points] Nash is PPAD-complete. State three properties about a PPAD graph. What does this tell you about the number of Nash equilibria in a game?

(d) [1 Points] Give a class of games for which the Nash problem can be solved efficiently.
4. **Auctions**

(a) **[2 Points]** Is the allocation rule of the first-price sealed bid auction monotonically non-decreasing (monotone)? Why or why not?

(b) **[2 Points]** Is the payment rule of the first-price sealed bid auction the “critical value” payment? Why or why not?

(c) Consider a FPSB auction for a single item, with two bidders with values $v_1 = 1$ and $v_2 = 2$. Bids can be in $b_1, b_2 \in [0, 10]$. If both bidders place the same bid then ties are broken in favor of bidder 2.

(i) **[2 Points]** Assume that bidders know each others’ values. What is a pure strategy Nash equilibrium in this auction?

(ii) **[2 Points]** Is this equilibrium unique? Justify your answer.

(d) **[1 Points]** What auction design sells to the highest bidder and uses the “critical value” payment?

(e) **[3 Points]** What are three differences between eBay’s auction design and the second-price sealed-bid auction?
5. **Peer prediction**

(a) **[3 Points]** Consider a setting with two possible signals \( \ell \) and \( h \). Suppose the output-agreement mechanism is used, paying $1 upon agreement and $0 otherwise. What needs to be true of \( \Pr(S_2 = h \mid S_1 = h) \) and \( \Pr(S_2 = \ell \mid S_1 = \ell) \) for the mechanism to be strict proper? **Provide a brief justification.**

(b) **[3 Points]** Suppose the distribution on signals by agents 1 and 2 is

\[
\begin{array}{ccc}
\text{Player 2} & h & \ell \\
\text{player 1} & h & 0.5 & 0.2 \\
& \ell & 0.2 & 0.1 \\
\end{array}
\]

Provide the payoff table for the 1/prior mechanism, and confirm that the mechanism is strictly proper.

(c) **[3 Points]** Your friend has designed a peer prediction method that he claims works on a subset of signals, such that

\[
\mathbb{E}_{s_2 \sim P(S_2|s_1=s_1)}[t_1(s_1, s_2)] > \mathbb{E}_{s_2 \sim P(S_2|s_1=r_1)}[t_1(r_1, s_2)], \quad \forall r_1 \neq s_1, r_1 \in S_1,
\]

for all \( s_1 \in S'_1 \subset S_1 \), but not for all signals \( s_1 \). The same property holds symmetrically for agent 2 on signals \( s_2 \in S'_2 \subset S_2 \). Will it be a strict best response for player 1 to report her signal truthfully when \( s_1 \in S'_1 \)? Justify your answer.
6. Prediction markets

(a) [2 Points] Consider the following bids and asks: bids 0.4, 0.3, 0.15, and asks 0.2, 0.25, 0.3. What is the range of clearing prices in a call market, and what would be the outcome if the midpoint of the range is used?

(b) [2 Points] Suppose a binary outcome, Clinton elected or Trump elected, and that your belief is $P(\text{Clinton}) = 0.6$. The current price on Clinton in the LMSR AMM is $0.7$. You will make a single trade, and make it now. Without providing a numerical analysis, how will you determine how to trade in the Clinton and Trump contracts?

(c) [3 Points] An internal prediction market that uses the LMSR AMM is used to predict two conditional outcomes:

- market success of the product given decision $A$ (current price $0.8$)
- market success of the product given decision $B$ (current price $0.4$)

The management will make decision $A$ or $B$ depending on which has the highest price in the prediction market (breaking ties in favor of $A$.) If decision $A$ is made then this contract pays $1$ if success and $0$ otherwise. All trading in the contract for the other decision is undone, and payments returned. Similarly if decision $B$ is made.

Your belief is $\Pr(\text{success given } A) = 0.8$ and $\Pr(\text{success given } B) = 0.7$.

Is there a new kind of strategic behavior in this “decision market”? Justify your answer, providing a non-technical argument that uses specifics of this example.
(d) Consider a combinatorial prediction market on the outcomes of all Harvard football games in 2016-17 season. For example, a trade can be made on a contract that pays out if a particular win/loss outcome occurs in each game.

(i) [1 Points] What is the problem with a CDA?

(ii) [2 Points] Give two problems with running the LMSR AMM for this application? Be brief.