1 Mechanism Design

1.1 Review

Key Points to Understand

1. Mechanism Design - where a system designer designs a mechanism which takes reports from self-interested agents (e.g. bids in an auction) and selects outcomes (i.e. allocation and payment) based on the agents’ reports. Some desirable characteristics: allocative efficiency, no deficit and participation.

2. Direct-revelation Mechanism (DRM)

   - (Mechanism Design without Money) : DRM is specified by an outcome rule $g : U \rightarrow O$
   - (Mechanism Design with Money) : DRM is specified by a choice rule $x : V \rightarrow A$ and a payment rule $t : V \rightarrow \mathbb{R}^n$

3. Strategy of agent $i$, $s_i : V_i \rightarrow V_i$.

4. A DRM is truthful (or strategyproof) if truthful reporting is a dominant strategy equilibrium (DSE).

5. (Revelation Principle) : Any function $f$ that can be implemented in a DSE of an indirect mechanism, can also be implemented in a DSE of a DRM.

4. VCG Mechanism : maximizes the total value of the agents and charges each agent the negative externality it imposes on the other agents by its presence. Formally, given reported valuation function $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$, the VCG mechanism is defined by

   - A choice rule $x(\hat{v}) \in \arg\max_{a \in A} \sum_{i \in N} \hat{v}_i(a)$
   - A payment rule $t$ s.t. $t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(a^{-i}) - \sum_{j \neq i} \hat{v}_j(a^*) \forall$ agents $i$

   where $a^* = x(\hat{v})$ and $a^{-i} = \arg\max_{a \in A^{-i}} \sum_{j \neq i} \hat{v}_j(a)$.

   - The VCG mechanism is truthful, efficient and satisfies participation.

5. Single Parameter Domain : each agent has a private input $w_i \in \mathbb{R}$, and a summarization function $q_i : A \rightarrow \mathbb{R}$ (known to the mechanism designer). For an allocation $a$, agent $i$-s value is $w_i \cdot q_i(a)$. Examples include single item (possibly multi-unit) auction. A mechanism $(x, t)$ is truthful for a single parameter domain if and only if, for all $i$ and $w_{-i}$
• (monotonicity) $q_i(x(\cdot, w_{-i}))$ is monotonically non-decreasing in $w_i$

• (payment identity) payment rule $t$ satisfies

$$t_i(w_i, w_{-i}) = w_i \cdot q_i(x(w_i, w_{-i})) - \int_{z=0}^{w_i} q_i(x(z, w_{-i})) dz$$

$$\hspace{2in} (1)$$

**Things you should be able to do**

1. Argue that a particular mechanism is / is not strategy-proof (reporting truthfully is a dominant strategy).

2. Given an allocation and a payment rule (perhaps both as formulas), be able to describe in words what is going on.

3. Given that in a VCG mechanism, each agent’s payment is the negative externality it imposes on the other agents by its presence, design the VCG mechanisms for particular allocation problems.

4. Illustrate that a VCG mechanism has / does not have the desirable characteristics of a mechanism, namely allocative efficiency, participation and no-deficit.

5. Given a monotone allocation rule, derive the payment using the payment identity for single parameter domains.

### 1.2 Practice

1. **VCG**

   Suppose that four bidders, Shosh, Matthew, Patrick and Perry are vying over a single good, for which they have values (20, 8, 15, 9), respectively.

   (a) What is VCG allocation and payment?

   (b) What is the second price sealed bid allocation and payment?

   (c) Now suppose that there are three identical goods and Shosh, Matthew, Patrick and Perry each want up to three goods and have values:

<table>
<thead>
<tr>
<th>Shosh</th>
<th>Matthew</th>
<th>Patrick</th>
<th>Perry</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st unit</td>
<td>20</td>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>2nd unit</td>
<td>6</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>3rd unit</td>
<td>0</td>
<td>5</td>
<td>11</td>
</tr>
</tbody>
</table>

   What are the VCG allocations and payments?

   **Answer :**

   (a) The VCG allocation would give the item to Shosh at a cost of 15 dollars.

   (b) The second price sealed bid allocation and payment are the same as in VCG.

   (c) Among all possible allocations, giving 1 unit to Shosh and 2 units to Patrick gives a total value of $20 + 15 + 15 = 50$ which is the maximum possible.

   Without Shosh, all units would be allocated to Patrick. Shosh’s payment would be $(15 + 15 + 11) - (15 + 15) = 11$. 
Without Patrick, 1 unit would be allocated to Shosh with value 20, 1 unit would be allocated to Perry with value 9 and the last unit would be allocated to Matthew or Perry with value 8. Therefore Patrick’s payment should be \((20 + 9 + 8) - 20 = 17\).

2. Consider the design of a mechanism for a simple double auction with one seller (agent 1), with a single item, and one buyer (agent 2). The outcome of the mechanism defines an allocation, \((x_1, x_2)\), where \(x_i \in \{0, 1\}\) and \(x_i = 1\) if agent \(i\) receives the item in the allocation, and payments \((t_1, t_2)\) by the agents. Let \(v_i\) denote the value of agent \(i\) for the item, and suppose quasilinear preferences, so that \(u_i(x_i, t_i) = x_i v_i - t_i\) is the utility of agent \(i\) for outcome \((x_1, x_2)\) and payment \((t_1, t_2)\).

(a) Specify the VCG mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies. [Don’t just give the general formula; write a succinct description for this application.]

(b) Provide a simple example to show that the VCG mechanism for a double auction runs at a deficit.

Answer: (a) Strategy space allows each agent to submit a bid (for the seller (agent 1) interpreted as minimum payment required and for the buyer (agent 2), interpreted as maximum willingness to pay) Let \(b_1, b_2 \geq 0\) denote bids for seller and buyer respectively.

If \(b_2 \geq b_1\) trade happens, and \(x_1 = 0, x_2 = 1\). Payments are

\[
t_1 = (\text{value of the buyer when seller is absent}) - (\text{value of the buyer when seller is present})
= 0 - b_2 = -b_2
\]

\[
t_2 = (\text{value of the seller when buyer is absent}) - (\text{value of the seller when buyer is present})
= b_1 - 0 = b_1
\]

Otherwise the seller keeps the item, and \(x_1 = 1, x_2 = 0\). Payments are

\[
t_1 = (\text{value of the buyer when seller is absent}) - (\text{value of the buyer when seller is present})
= 0 - 0 = 0
\]

\[
t_2 = (\text{value of the seller when buyer is absent}) - (\text{value of the seller when buyer is present})
= b_1 - b_1 = 0
\]

(b) To see VCG in double auction can run at a deficit, consider \(v_1 = 5, v_2 = 10\). Then total deficit will be \(-10 + 5 = -5\).
3. A sealed-bid auction for a single item is defined in terms of a choice rule \( x : V \rightarrow A \) and payment \( t : V \rightarrow \mathbb{R}^n \). For a given valuation profile \( v \), \( x_i(v) = 1 \) if agent \( i \) gets the item and it is 0 otherwise. For any such auction \((x,t)\), we are interested in two properties:

(A1) \textbf{(Agent-Independent payment)}: For every \( v_{-i} \) there does not exist two different reports \( v_i, v'_i \) of agent \( i \) such that the same outcome is selected but \( t_i(v_i, v_{-i}) \neq t_i(v'_i, v_{-i}) \). In other words, for all \( v_i \), the payment is independent of \( v_i \) as long as the same outcome is chosen. Such a payment rule can be described by price functions

\[
p_i : \{0,1\} \times V_{-i} \rightarrow \mathbb{R}
\]

(A2) \textbf{(Agent-Optimizing allocation)}: For every agent \( i \), allocation rule \( x_i(v) \in \{0,1\} \) selects an allocation

\[
x_i(v) \in \arg \max_{z \in \{0,1\}} \{ z \cdot v_i - p_i(z, v_{-i}) \}
\]

where \( p_i \)-s are the price functions as defined in (A1)

(a) Prove that any auction that satisfies A1 and A2 is truthful in a dominant-strategy equilibrium.

(b) Define the second-price Vickrey auction in these terms (i.e. exhibit an agent-independent price function (A1) and demonstrate that the winner determination rule satisfies (A2).)

\textbf{Answer}:

(a) Suppose otherwise, i.e. there is some \( v_{-i} \) and \( v'_i \neq v_i \) for which agent \( i \) benefits by reporting \( v_i \) instead of \( v'_i \), i.e.

\[
z' \cdot v_i - p_i(z', v_{-i}) > z \cdot v_i - p_i(z, v_{-i})
\]

where \( z = x_i(v_i, v_{-i}) \) and \( z' = x_i(v'_i, v_{-i}) \). But \( z \in \arg \max_{z \in \{0,1\}} \{ z \cdot v_i - p_i(z, v_{-i}) \} \) and this is a contradiction.

(b) For a Vickrey auction, define \( p_i(0, v_{-i}) = 0, p_i(1, v_{-i}) = \max_j \neq i v_j \), and to establish (A2) see that if \( v_i > \max_j \neq i v_j \) then the agent wins and if \( v_i < \max_j \neq i v_j \) then the agent loses.
4. Low-cost Routing

(a) Take the above buying-a-path-in-a-network problem. Each edge corresponds to an agent $i$ with cost $c_i > 0$ if the edge is used. Bob wants to get from S to F so he asks for path costs of all the agents and runs the VCG mechanism described in the notes. Which path does Bob purchase, and what are the VCG payments for the agents along the path?

(b) Bob implements the mechanism and takes the optimal path and pays the correct VCG amount. But now Bob realizes that he wants to travel back home, from F to S. He could use VCG again, but he can actually do better. What are some alternative path-procurement mechanisms that are better for Bob than using VCG again?

(c) Find an example graph where the total VCG payment for the best path is arbitrarily greater than the cost of the path.

**Answer:**

(a) With VCG, we find the cheapest path and then calculate payments along this path. The cheapest path is d-e, with a total cost to the owners of 35.

Now we calculate payments. Without d, the cheapest path is g-f-e, which costs 40. So the edge d pays $(-40) - (-15) = -25$. In other words, edge d is compensated 25.

Without e, the cheapest path is a-b. This costs 50. So e pays $(-50) - (-20) = -30$. In other words, e is compensated 30.

In total, Bob pays 55 and travels along d and e.

(b) Bob just ran the VCG mechanism, so he knows the true costs of all the edges. He could certainly run VCG again and pay 55, but perhaps Bob could negotiate prices with a and b. Bob could offer to pay a 21 and b 31, giving him a total cost of 52, which is better than the cost of 55 he paid through VCG. Perhaps Bob could even negotiate lower prices with d and e. Bob can offer to pay d 21 and e 16 and threaten to use path a-b in case d or e rejects the prices. This incurs Bob an even lower cost of 37.
Bob was not able to implement this negotiation in part (a) because he did not know the costs of any paths.

(c) VCG selects the lower path, but payments to a and b are each $100,000,000,000,000$.

5. TV Advertising

Consider an auction where advertisers are bidding for Superbowl commercials. There are spots for three commercials to air and the issuer with the first spot gets 40 views, the second gets 25 views, and the third gets 10 views (numbers in millions). Each advertiser wants one of the three commercials and only cares about the number of views she gets.

(a) Assume a mechanism designer runs an auction where bidders bid their willingness to pay per view. Come up with a choice rule that is monotone non-decreasing in every report $\tilde{v}_i$.

(b) Consider the simple monotone non-decreasing choice and payment rules where the highest bidder receives the top spot and pays the second highest price per view; the second highest bidder receives the second spot and pays the third highest price per view, and the third highest bidder receives the third spot and pays the fourth highest price per view. Is this mechanism truthful? If so, show that it is truthful, if not come up with an example where it is not truthful.

(c) Come up with a payment mechanism that is truthful using the payment identity from the reading on mechanism design (eq. 1). Suppose there are four advertisers participating in the auction and the bids of the first three agents are 4, 6, and 10. Draw the allocation vs value curve and the utility vs value curve for a fourth agent with valuation $v_4 = 20$.

Answer: (a) The highest bidder receives the top spot, the second highest receives the second spot, and the third highest receives the third spot. Break ties by awarding the higher spot to the agent with the lower index.

(b) No. This mechanism is not truthful. Imagine a situation where four agents have valuations per view of 20, 19, 2, and 1. Under this mechanism, the player who bids 20 pays 19 on 40 views, giving him a total utility of $(20 - 19) \times 40 = 40$. However, if instead he were to bid 18, he would come in second, pay 2 and receive 25 views, giving him a total utility of $(20 - 2) \times 25 = 450$.

(c) First note that, given a valuation profile $v$ of the four agents, the allocation $x(v)$ is a four-dimensional vector where $x_i(v)$ is the slot allocated to ad $i$. For example, $x_4(v) = 1$ if ad 4 gets
the first spot, 2 if she gets the second spot, 3 if she gets the third spot and 0 otherwise. Since
the advertiser is only interested in number of views of an allocation, the summarization function
maps the allocation to the corresponding number of views i.e.

\[
q_4(x(v)) = \begin{cases} 
40 & \text{if } x_4(v) = 1 \\
25 & \text{if } x_4(v) = 2 \\
10 & \text{if } x_4(v) = 3 \\
0 & \text{if } x_4(v) = 0 
\end{cases}
\]

Now, we want to use the payment identity. To start thinking about this, we draw a graph with
the summarization function \( q_4(x(v_1, v_4)) \) along the y-axis and \( v_4 \) of our fourth agent along the
x-axis.

The payment identity gives us

\[
t_4(v_4, (4, 6, 10)) = v_4 \cdot q_4(x(z, v_4)) - \int_{z=0}^{z=v_4} q_4(x(z, v_4))dz.
\]

For a given value of \( v_4 \), the agent’s utility is \( v_4 \cdot q_4(x(v_4, v_4)) - t_4(v_4, (4, 6, 10)) \) which is nothing
but the area under the allocation curve from 0 to \( v_4 \). The shaded area in the next graph shows
the agent’s utility if his valuation is 12.
From the graph we can express utility of agent 4 as a function of $v_4$.

$$u_4(v_4, v_{-4}) = \begin{cases} 
0 & \text{if } 0 \leq v_4 \leq 4 \\
10(v_4 - 4) & \text{if } 4 < v_4 \leq 6 \\
20 + 25(v_4 - 6) & \text{if } 6 < v_4 \leq 10 \\
120 + 40(v_4 - 10) & \text{if } v_4 > 10 
\end{cases}$$

The below graph illustrates his utility plotted against all values of $v_4$.

# Online Ad Auctions

**Key Points**

(a) Three types of ads were discussed in class:

- **display ads**: pre-approved advertisers (like Marc Jacobs) bid (in real-time) to display some graphical advertisement on each load of a provider’s (the NYT) page through an Ad exchange (like AdX).
• **sponsored search**: advertisers bid on search-terms ahead of time (with *standing bids*). Each time a search term is entered, the search engine runs a GSP auction with eligible advertisers. Payments are usually made per-click.

• **contextual ads**: Generally keyword-based standing bids like sponsored search, only ads appear on external webpages that users visit, and are based on the content of the webpage.

(b) In the **Generalized Second-Price (GSP)** auction, advertisers submit one bid each for multiple slots.

- **Assumptions**: Advertisers $i$ have uniform per-click values $v_i$, independent of position, number of clicks, etc. Each advertiser has an ad quality $q_i$ (calculated by the auction designer). Each ad position $j$ has a position value $p_j$ and each ad’s click-through rate is modeled by $CTR_{ij} = q_i \cdot p_j$. Effective values are given by $v_{ij} = CTR_{ij} \cdot v_i$.

- **Allocation rule**: Advertisers are ranked into slots according to decreasing $b_i q_i$, where $b_i$ is the $i^{th}$ advertiser’s bid per click.

- **Payment rule**: Advertisers pay the least they would have needed to bid to keep their slots.

$$PPC_{gsp,i} = \frac{q_{i+1} b_i + 1}{q_i}$$

where we assume that indices are ordered by effective bid order.

- The GSP auction is not truthful. However, **balanced bidding** is a Nash equilibrium in GSP. Balanced bidding is defined by each advertiser bidding just high enough such that the next-highest advertiser bidding down to switch slots would not decrease the original advertiser’s utility. That is,

$$p_{i-1}(v_i - b_i) = p_i(v_i - b_{i+1})$$  (3)

for each advertiser $i \geq 2$ and $b_1 \geq b_2$.

(c) The **VCG position auction** ranks the advertiser according to decreasing $q_i \cdot b_i$. The expected payment by advertiser $i$ is given as

$$t_{vCG,i}(b) = \sum_{k=i+1}^{n} (p_k - p_{k+1})q_kb_k$$

and the per click payment for advertiser $i$ is $t_{vCG,i}(b)/(q_i \cdot p_i)$. The VCG position auction is strategyproof and allocatively efficient. In fact, the balanced bidding outcome in the GSP auction is the same as the truthful outcome in the VCG auction. Search engines still use GSP though (why?)

### 2.1 Practice

1. (GSP.) Consider the following position auction setting. Suppose our auction is a GSP auction with three positions and four bidding advertisers A, B, C, and D. The three positions have position effects $p_1 = 0.3, p_2 = 0.28, p_3 = 0.1$. The bidders have per-click values (e.g. the customer value upon the click) $v_A = $100/click, $v_B = $50/click, $v_C = $18/click, and $v_D = $10/click.
(a) Suppose that the advertisers create ads with qualities \( q_A = 0.5, q_B = 0.7, q_C = 0.4, q_D = 0.8 \). If advertisers bid truthfully (i.e. bid \( b_i = v_i \)), what will the outcome be (i.e. what are the position allocation and payment for each advertiser)?

**Answer:**

Allocations: ordering by \( b_i q_i \), we get \( x_A = 1, x_B = 2, x_C = \text{unallocated}, x_D = 3 \).

Payments: each bidder pays \( q_i + 1 + b_i q_i \), so \( PPC_A = 70, PPC_B = \frac{80}{7}, PPC_C = 0, PPC_D = 9 \).

(b) From now on suppose that all advertisers have equal qualities \( q_i = 1 \) for all \( i \in \{A, \ldots, D\} \).

**Answer:**

Allocations: \( x_A = 1, x_B = 2, x_C = 3, x_D = \text{unallocated} \). Payments: \( PPC_A = 50, PPC_B = 18, PPC_C = 10, PPC_D = 0 \).

(c) Is this truthful bidding a Nash equilibrium?

**Answer:**

No, let advertiser A deviate and bid 30. Then advertiser A’s new expected utility \( u_A' = p_2(v_A - b_3) = 0.28(100 - 18) > 0.3(100 - 50) = p_1(v_A - b_2) = u_A \).

(d) Suppose that we have bids \( b_A = 20, b_B = 50, b_C = 18, b_D = 10 \). Is this a Nash equilibrium? Is this outcome value-ordered (i.e. are the bids non-decreasing in the values)?

**Answer:**

Yes NE. \( u_A = 0.28(100 - 18) = 22.96, u_B = 0.3(50 - 20) = 9, u_C = 0.1(18 - 10) = 0.8, u_D = 0 \). Check deviations for each advertiser. No, not value-ordered.

(e) The bids not being value-ordered in Nash equilibrium implies that this bidding outcomes does not satisfy the balanced-bidding condition. Check that the balanced-bidding condition is violated for some ad \( i \).

**Answer:**

In particular for ad A allocated at slot 2, we need \( p_1(v_A - b_A) = p_2(v_A - b_C) \), but \( 0.3(100 - 20) > 0.28(100 - 18) \). This also implies that, advertiser A could bid higher and still not risk retaliation from advertiser B that would hurt him.

(f) What is the balanced bidding outcome in this example? For this to be Nash, what is the key assumption we must make about the knowledge of the bidders?

**Answer:**

\( b_D = v_D = 10 \). Now \( b_C \) satisfies \( p_3(v_C - b_D) = p_2(v_C - b_C) \Rightarrow b_C = \frac{106}{15} \). Similarly \( p_2(v_B - b_C) = p_1(v_B - b_B) \Rightarrow b_B = \frac{262}{15} \). Pick any \( b_A = v_A = 100 \) and we’re done. The key assumption we make is that each bidder knows the others’ true values.

(g) Check that the balanced bidding outcome is Nash.

**Answer:**

Yes. \( u_A = 0.3(100 - \frac{262}{15}) > 0.28(100 - \frac{106}{15}) \) and similarly any lower deviation. Also \( u_B = 0.28(50 - \frac{106}{15}) > 0.1(50 - 10) \), and B would never try to outbid A. The check for C is similar.