Total points: 68. This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of answers, and we encourage typed submissions. You are free to discuss the problem set with other students but you must not share your answers. Extra credit will only be considered as a factor in deciding the letter grade for the course at the end of the term.

1. [20 Points] Nash equilibrium

   (a) [3 Points] Prove Theorem 2.1. Be sure to argue both the necessary (“only if”) and sufficiency (“if”) directions.

   (b) [2 Points] What strategies survive iterated elimination of strictly-dominated strategies in this normal-form game?

   \[
   \begin{array}{c|ccc}
   & L & C & R \\
   \hline
   T & 2,0 & 1,1 & 4,2 \\
   M & 3,4 & 1,2 & 2,3 \\
   B & 1,3 & 0,2 & 3,0 \\
   \end{array}
   \]

   (c) [4 Points] What are the pure strategy Nash equilibria of the game? Find a non-trivial (i.e., with support of two or more actions) mixed-strategy NE.

   (d) [5 Points] Prove that iterated elimination of strictly-dominated actions never removes an action that is in the support of any mixed-strategy Nash equilibrium.

   (e) [1 Points] Give an example of a game where no action can be eliminated by iterated elimination of strictly-dominated actions.

   (f) [5 Points] Explain why the time complexity of iterated elimination of strictly-dominated actions by pure strategies (don’t worry about mixed strategies) is \(O(m^{n+2}n^2)\) for \(n\) players, each with \(m\) actions.

2. [12 Points] Scheduling game

Consider a game that involves scheduling tasks onto shared machines. Each agent \(i \in \{1, 2, 3\}\) has a task to complete and chooses a machine. Let \(c_{ij}\) denote the time the task of agent \(i\) takes on machine \(j\).

The costs are: \(c_{11} = 12, c_{12} = 10, c_{21} = 16, c_{22} = 10, c_{31} = 2, c_{32} = 16\), so that machine 2 is faster than machine 1 for jobs 1 and 2, but not for job 3. Each machine has a precedence
order, determining the sequence it will work on assigned tasks. The cost to an agent is the time when its own task completes, and an agent wants to minimize its cost.

Each machine adopts a shortest-first precedence order, preferring tasks that are shorter, and breaking ties in favor of agents with a lower index.

(a) [4 Points] State the precedence order on tasks for each machine. Give a pure-strategy Nash equilibrium, and argue why it is an equilibrium.

(b) [4 Points] Explain without enumerating all possible action profiles why the Nash equilibrium identified in part (a) is unique.

(c) [4 Points] The make-span is the latest time that any task is completed. What is the make-span in this Nash equilibrium? What is the socially optimal assignment; i.e., the one that minimizes the make-span? Justify your answer.


Consider a variation on the Bargaining game, in which the payoffs to the players after action me, even and you from player 1 and “yes” from player 2 are (2, 0), (1, 1) and (0, 2), respectively. The payoffs remain (0, 0) if player 2 responds with “n” to any offer.

(a) [4 Points] Write out the normal-form representation, and determine the set of pure strategy Nash equilibria

(b) [3 Points] Determine the set of subgame-perfect equilibria. What is it about this variation on the Bargaining game that yields multiple subgame-perfect equilibria?

(c) [3 Points] Change the payoffs in response to “n” from player 2 (using the same payoffs after “n” everywhere in the game) so that even and y is played in the equilibrium of the unique subgame-perfect equilibrium.

4. [14 Points] Infinitely repeated Prisoners’ Dilemma

(a) [10 Points] Prove that Tft is not a subgame-perfect equilibrium of the infinitely repeated Prisoners’ Dilemma when adopted by both players, for any discount factor $\delta < 1$. [Hint: there are four classes of strategically equivalent subgames, and you will show that there are two classes of subgames for which there is no $\delta$ that simultaneously precludes a profitable single-period deviation in both classes.]

(b) [4 Points] Establish the range of discount factor $\delta$ for which the revised grim trigger in Figure 4.10 is a subgame-perfect equilibrium when adopted by both players. For this, follow the analysis approach in the proof of the Nash-Threat Folk theorem (Theorem 4.8). Give a brief, non technical explanation of why the modification from the grim trigger in Figure 4.7 is important in obtaining a subgame-perfect equilibrium.

5. [12 Points] Correlated equilibrium

(a) [3 Points] Give a normal-form game (different from the one in Figure 2.6) where there is a correlated equilibrium that is not just a distribution on pure-strategy Nash equilibria. Justify your answer.
(b) [3 Points] Explain how any correlated equilibrium that is a distribution on pure-strategy Nash equilibria can be realized through a public signal (visible to all agents), and strategies that select for each agent an action that depends on the signal. Explain why the strategy profile is an equilibrium.

(c) [3 Points] Now consider a more general way to think about correlated equilibria, as described through the use of joint distribution \( p^*(a) \) on action profiles (see Definition 2.11). Give a simple interpretation of this equilibrium in terms of a random signal, that is correlated but private to each agent, and a strategy for each agent that is a mapping from its signal into an action.

(d) [3 Points] Prove that the correlated equilibria that are defined for the game of Chicken in Example 2.7 (p. 28) are Pareto optimal distributions (see Section 2.1.3) on action profiles, for any distribution on the value of the signal.
6. **Extra credit (see syllabus for explanation)**

(a) Provide a precedence ordering for machines 1 and 2 in the scheduling game in Problem 2 for which there is no pure-strategy Nash equilibrium. Explain.

(b) Prove that \text{TFT} is a Nash equilibrium of the infinitely repeated *Prisoners’ Dilemma* when adopted by both players, for some discount factor \( \delta < 1 \). (Caution, TFT is not a subgame-perfect equilibrium, and so you will not be able to show this using the single-deviation principle.)

(c) Prove that the automaton strategy in Figure 4.11 forms a Nash equilibrium of the infinitely repeated *Prisoners’ Dilemma* when adopted by both players, for a suitably large \( \delta < 1 \), and with lexicographic preference for simplicity. Trace the behavior of this strategy when adopted by both players.