1 Normal Form Games

Game theoretic models are used to study complicated strategic situations in a compact, digestible
form that can be analyzed rigorously. At a high level, game theoretic models break down situations
into sets of decision problems for independent agents: Given circumstance $A$ that involves actors
$b$ and $c$, each of whom have choices $x, y, z$, what will the actors choose to do if the consequences of
their actions depend on both of their choices? Game theoretic models generally assume rational
agents who always choose their strategies so as to maximize their happiness, as represented by a
utility function. While this may seem very restrictive, the choice of utility function can be quite
flexible, and make agents selfish, altruistic, and otherwise. The choice of model is often the most
interesting and difficult part of game theory. For now, however, we’ll focus on the anatomy of a
game.

1.1 Review

A simultaneous-move game is defined by:

1. $N = \{1, \ldots, n\}$ agents indexed by $i$

2. $A = A_1 \times \ldots \times A_n$, where $A_i$ is a set of actions available to agent $i$ and $a = (a_1, \ldots, a_n) \in A$
denotes an action profile

3. A payoff map or utility function that maps each combination of strategies to a utility for
each player.

A normal form representation of a simultaneous-move game is a payoff matrix that describes $A$.

\footnote{For example, in a 2-player game in which each player has $m$ possible actions, $A$ may be represented as an $m \times m$
matrix in which each row corresponds to an action that player 1 can take, each column corresponds to player 2’s
actions, and the $ij$’th entry of the matrix is a tuple $(u_1(a_i^1, a_j^2), u_2(a_i^1, a_j^2))$ with the payoff that players 1 and 2 get
when actions $i$ and $j$, respectively, are taken.}
1.2 Practice

Consider the following TF-Student Game:

Each week, a CS136 TF has to choose whether to put a lot of effort, little effort, or no effort at all into preparing section. The students, on the other hand, must decide whether to attend section or not (let’s model the students as one player here). Preparing section is costly for the TF, but if many students come and he is well-prepared, he gets high reward. If he is poorly prepared and students attend section, he will be embarrassed. The students’ utility depends on how well the TF is prepared. If the students don’t attend section, they get 0 utility.

1. Write down the normal form of this game with reasonable utilities.

2. What (is/are) the Pure Strategy Nash Equilibri(um/a) of this game?

2 Solution Concepts: Best Response, Dominant Strategies and Nash Equilibria

2.1 Review

A solution concept is a formal rule for predicting how a game will be played. Here are some basic solution concepts:

1. Best Response: Player $i$’s best response strategy to a strategy profile (tuple of strategies) $a_{-i}$ of the other agents is his best option under the assumption that all other players will play according to $a_{-i}$. In other words, given the strategies of players $-i$, player $i$’s best response is a strategy that gives him the most utility. Remember that an action profile is a Nash Equilibrium if each player is choosing his best response to each other player.

2. Strictly Dominated Strategy: A strategy $a_i$ for player $i$ strictly dominates strategy $a_i'$ if $u_1(a_i, a_{-i}) > u_1(a_i', a_{-i}) \forall a_{-i}$. (Note: $a_{-i}$ refers to all agents except $a_i$, so in fact $\{a_i, a_{-i}\}$ is a vector of the actions of all agents.)

3. Iterated Elimination of Strictly Dominated Strategies: No player will ever choose a strictly dominated strategy, so we can iteratively eliminate dominated strategies.

Given remaining (not yet eliminated) actions $R_1 \subseteq A_1,..., R_n \subseteq A_n$ for each agent, we look for an action for player $i$ that is strictly dominated by another action.
This means no dominated strategy will ever be used in either a Pure Strategy or Mixed Strategy NE.

4. **Nash Equilibrium (NE):** In a n-player game, strategy profile \((s^*_1, \ldots, s^*_n)\) is a *Nash equilibrium* if \(u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \forall s_i \forall \text{ players } i\). In other words \((s^*_1, \ldots, s^*_n)\) is a NE if \(s^*_i\) is a best response to \(s^*_{-i} \forall \text{ players } i\). Nash equilibria can be *pure strategy Nash equilibria (PSNE)* (pure strategies only) or *mixed strategy Nash equilibria (MSNE)*.

### 2.2 Practice

For this exercise, consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>(3,0)</td>
</tr>
<tr>
<td>Player 1</td>
<td>Middle</td>
</tr>
<tr>
<td></td>
<td>Down</td>
</tr>
</tbody>
</table>

1. What is the best response of player 1 when player 2 plays Right? What is the utility she gets?

2. Apply iterated elimination of dominated strategies to this game. Which strategy profiles remain, and what are the corresponding utilities? What is (are) the NE(s)?

Now consider a different game:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>(-1,3)</td>
</tr>
<tr>
<td>Player 1</td>
<td>Middle</td>
</tr>
<tr>
<td></td>
<td>Down</td>
</tr>
</tbody>
</table>

3. Can you iteratively eliminate any dominated strategies?

4. Find all NEs of this game. (Hint: Wilson’s Theorem (1971) says that all non-degenerate finite games have an odd number of NE)

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2Note that this is different from iterated elimination of *weakly* dominated strategies. To see how IEWDS could lead to problems, see [here](#).
3 Correlated equilibrium

3.1 Review

A correlated equilibrium is a different equilibrium concept for games where the play can be represented as a joint distribution. A Nash equilibrium is also a correlated equilibrium, representing the special case where actions are independently sampled. Any distribution over the set of Nash equilibria of a game is a correlated equilibrium. However, the set of correlated equilibria is richer, and not restricted to mixtures over Nash equilibria.

A probability distribution \( p^* \) on action profiles \( A \) is a correlated equilibrium in a simultaneous-move game if and only if the expected utility to agent \( i \) for playing action \( j \), conditioned on the event that \( j \) is drawn from joint distribution \( p^* \), is at least as much as the expected utility from playing any other action.

\[
\sum_{y \in A_{-i}} u_i(j, y) \cdot p^*_i(y | j) \geq \sum_{y \in A_{-i}} u_i(j', y) \cdot p^*_i(y | j)
\]

for all actions \( j \) with \( p^*_i(j) > 0 \), all actions \( j' \in A_i \), and all agents \( i \).

3.2 Example

Consider the Chicken game below.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Straight</td>
</tr>
<tr>
<td>Straight</td>
<td>(-5, -5)</td>
</tr>
<tr>
<td>Player 1</td>
<td>(-3, 2)</td>
</tr>
</tbody>
</table>

1. Find the two pure strategy Nash equilibria.

2. Find the mixed strategy equilibria and compute the expected payoff.

3. Explain why the uniform distribution over the action profiles \{ (CC), (SC), (CS) \} is a correlated equilibrium.

4 Congestion Games

4.1 Introduction

The normal form representation becomes unwieldy as the number of players grows. Imagine we had a prisoner’s dilemma except there were 1000 prisoners each with the choice to either cooperate or defect. How many numbers would it take to write down the payoff matrix in that case? In general, for \( n \) players with \( m \) actions each?

For some game types, such as the congestion game (Chapter 2.8.1), we can find more succinct ways to represent the game. A congestion game is a simultaneous-move game in which there is a set of
resources $R$, and each agent selects one or more resources. Each resource has an associated cost
that depends only on the number of agents using it (not on the identities of those agents).

An example of a congestion game is rush hour in Manhattan when multiple agents are trying to
drive from one part of the city to another. They have a number of different routes to choose from,
and the cost to traveling on any particular segment of road is some function of the number of other
cars using that same road. Each player wants to complete her commute while minimizing the total
cost of the trip. Try writing up a reasonable model of rush hour using the notation for a congestion
game below.

4.2 Notation

A congestion game $(N, R, A, c)$ is defined by:

1. Set of players $N = \{1, \ldots, n\}$
2. Set of resources $R = \{1, \ldots, m\}$
3. Joint action set $A = A_1 \times \cdots \times A_n$, where $A_i \subseteq 2^R$ is the set of actions available to player $i$. Note that in general not all actions have to be available to all players.
4. Cost function $c_j(x) : \{0, \ldots, n\} \to \mathbb{R}$ for resource $j$ which depends only on the number of agents $x$ that use resource $j$

Congestion games have the following properties:

1. Total cost to agent $i$: Given action profile $a = (a_1, \ldots, a_n) \in A$ which leads to $x_{j,a}$ players using resource $j$ for each $j \in R$, player $i$’s cost is $c_i(a) = \sum_{j \in a_i} c_j(x_{j,a})$.
2. Existence of pure strategy Nash equilibrium: Congestion games always have a PSNE (see chapter 3 for a sneak peek at a proof).

4.3 Practice

Consider the following Party or Pset Game:

On Friday night, the 100 students in CS136 each have the option of going to a CS136 party or
psetting. For each student, psetting yields cost $-1600$ (note that a negative cost corresponds to a
positive utility: who doesn’t enjoy the reward of finishing a 10-hour assignment!), while going to
the party yields cost $-2500 + (x - 50)^2$, where $x$ is the total number of students at the party. The
party is exclusively for CS136 students, so 100 students will attend at maximum.

1. How should we write this game in normal form? How big would that representation be?

2. How can we model this as a congestion game (i.e. what are $(N, R, A, c)$)?

3. What is a PSNE for this game? (Hint: graph a student’s utility against the number of
partygoers for each of the 2 pure strategies.)
This example illustrates that congestion games need not model “congestion” in the traditional sense, and only rely on the anonymity of players and shared “resources” whose values depend solely on the number, not identities, of players using them.