1. Game theory I

Consider the repeated Stag hunt game:

\[
\begin{array}{c|cc}
\text{Player 1} & S & H \\
\hline
S & 4, 4 & 0, 1 \\
H & 1, 0 & 1, 1 \\
\end{array}
\]

(a) Give an example of a subgame-perfect Nash equilibrium strategy in an infinitely repeated Stag hunt game in which the equilibrium play includes both S and H. Briefly justify your answer.

(b) Modify the payoff for one of the action profiles so that, when repeated a finite number of times, the game has a unique subgame-perfect Nash equilibrium. Briefly justify your answer.
2. Game theory II

Consider the following game.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>(3,4)</td>
</tr>
<tr>
<td>Center</td>
<td>(4,6)</td>
</tr>
<tr>
<td>Down</td>
<td>(2,3)</td>
</tr>
</tbody>
</table>

(a) Which strategies can be eliminated through iterated eliminated of strictly dominated strategies?

(b) Using only the remaining strategies, solve for the pure strategy Nash equilibrium/equilibria or argue that none exist.

(c) Solve for the mixed strategy Nash equilibrium.
(d) Suppose now the game is played as a sequential game. If you are player 1, would you prefer to move first or second, or are you indifferent?
3. Game theory III

In the following network routing game, each of four agents has to route a unit flow from its source to sink, and each edge has a delay (or cost) that depends on the total flow on the edge. For example, edge $1 \rightarrow 2$ has linear cost $x_{12}$, which is the total flow on the edge, while edge $2 \rightarrow 1$ has cost 0. Each agent wants to minimize the total delay on its selected path from source to sink, which is the sum of the delay on each edge.

(a) Formulate this as a congestion game – give the resources, the action sets, and the cost functions.

(b) Give two pure-strategy Nash equilibria of the game and explain why each one is an equilibrium.

(c) What is the socially optimal flow?
4. Mechanism Design I

Consider a multicast routing problem. A server at $A$ can transmit content to nodes $D, E, F$ and $G$. At each node there may be 1 or more users. User $i$ has value $w_i > 0$ for service. To receive service the edges between the user and the server must be activated. Each edge has an associated cost, as illustrated in the figure.

In the mechanism design problem, the edge costs and network topology are known, but the values of users are private. The outcome of a mechanism defines the set of activated edges and the payment by each user.

Suppose the network operator uses the VCG mechanism to elicit values from users.

(a) What does the outcome rule of the VCG mechanism do in this context? [Just in words.]

(b) Briefly, is truthful reporting a dominant-strategy equilibrium for users? Why or why not?

(c) What is the outcome of the VCG mechanism for truthful reports?

(d) Do the total payments cover the total edge cost incurred by the network?
(e) Modify the value of the user at $D$ in a way that the total payments in the VCG outcome will cover the costs (when users are truthful).

(f) Can users 3 and 4 collude and both submit high bids, receive service, and make no payment? Justify your answer.
5. Algorithmic Game Theory

(a) Explain how to represent a congestion game in which each agent can choose a single resource as a (succinct) action-graph game.

(b) Pure-strategy Nash equilibrium can be computed in polynomial time in the size of the normal-form representation of a game. Give two reasons for why the impact of this on algorithmic game theory is limited?

(c) Consider a joint probability distribution $p^*$ on action profiles $a \in A$, with $p^*(a_{-i} \mid a_i)$ to denote the conditional probability of action profile $a_{-i}$ given action $a_i$, and $p^*(a_i) = \sum_{a_{-i} \in A_{-i}} p^*(a_i, a_{-i})$ to denote the marginal probability of $a_i$. What is wrong with (1) in the following statement (there is one problem)?

$p^*$ is a correlated equilibrium if, for all agents $i$, and all actions $a_i \in A_i$ with $p^*(a_i) > 0$, we have:

$$\sum_{a_{-i} \in A_{-i}} p^*(a_{-i} \mid a_i)u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} p^*(a_{-i} \mid a'_i)u_i(a'_i, a_{-i}), \quad (1)$$

for all actions $a'_i \in A_i$.

(d) Nash is PPAD-complete. State three properties about a PPAD graph. What does this tell you about the number of Nash equilibria in a game?
(e) State a family of games for which the Nash problem can be solved efficiently.
6. **Auction design I**

(a) Is the allocation rule of the FPSB auction monotone? Why or why not?

(b) Is the payment rule of the FPSB auction the “critical value” payment? Why or why not?

(c) Imagine applying the revelation principle to the first-price sealed bid auction under uniform value distributions. Is the auction truthful? Is the auction allocatively-efficient? Justify your answer.

(d) Consider a FPSB auction for a single item, with two bidders with values \( v_1 = 1 \) and \( v_2 = 2 \). Bids can be in \( b_1, b_2 \in [0, 10] \). If both bidders place the same bid then ties are broken in favor of bidder 2.

(i) Assume that bidders know each others’ values. What is a pure strategy Nash equilibrium in this auction? [Note: this is asking for a Nash eq., not a BNE.]

(ii) Is this equilibrium **unique**? Justify your answer.

(e) What auction design sells to the highest bidder and uses the “critical value” payment?
(f) What are three differences between eBay’s auction design and the second-price sealed-bid auction?
7. Revenue optimizing auctions

(a) For distribution function $G_i$ and density function $g_i$, the virtual value function is

$$\phi_i(v_i) = v_i - \frac{1 - G_i(v_i)}{g_i(v_i)}$$

Derive $\phi_i(v_i)$ for $v_i \sim U(a,b)$, for $a \geq 0, b > a$.

(b) Consider $v_1 \sim U(0,2), v_2 \sim U(1,3)$. Plot the v.v.s for bidder 1 and 2 on the same graph.

(c) Consider the Myerson auction. Suppose $b_2 \leq 1.5$. What is the range of bids of bidder 1 for which bidder 1 wins, and what is her payment?

(d) Consider the Myerson auction. Suppose $b_2 = 3$. What is the highest payment bidder 2 might make? How does this compare with the SPSB auction?
(e) Prior-free auctions: Give an informal argument for why the DOP auction is strategy-proof.

(f) Consider values 30, 14, 9, 5 and truthful bids. What is the target revenue $R_{opt}^{(2)}$, and what is the outcome of DOP?

(g) Use a variation on the example in part (f) to explain what goes wrong with DOP in regard to the $R_{opt}^{(1)}$ target.
8. **Alg. Mechanism Design**

Consider a *greedy, single-minded combinatorial auction* with bids \((\hat{w}_i, \hat{T}_i)\) from each bidder (bid value, target bundle):

1. Rank bids by decreasing score \(\sigma(\hat{T}_i, \hat{w}_i)\) (break ties arbitrarily)
2. Allocate packages greedily in order of decreasing score, charge the critical value payment.

(a) Briefly, why is the VCG mechanism not well suited to this problem of algorithmic mechanism design?

(b) Briefly, what fails about the greedy auction if score \(\sigma(\cdot, \cdot)\) is not increasing in \(\hat{w}_i\) or decreasing in \(|\hat{T}_i|\)?

(c) Consider score function \(\sigma(\hat{w}_i, \hat{T}_i) = \hat{w}_i\), and bids \((A, 10), (BC, 8), (AB, 7), (AD, 6), (B, 5), (C, 5)\).

   step 1 What is the efficient allocation?

   step 2 What is the allocation and payments in the greedy auction? Is the auction strategy-proof?

   (i) What is the outcome in VCG-based using the same greedy algorithm? What do you notice? Also give a useful manipulation by a losing bidder.