• Gibbard Satterthwaite impossibility
  – 3 or more outcomes
  – All strict preference orders possible
  – onto and strategy-proof => dictatorial
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• “strategy-proof mechanisms are uninteresting”

• What to do?
  – Restrict preference domain
  – Two outcomes
  – Not SP
• Gibbard Satterthwaite impossibility
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• What to do?
  – **Restrict preference domain**
  – Two outcomes
  – Not SP

---

Facility location (not matching!)
(Moulin'80)

(Restrict to "single-peaked" preferences)
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(Moulin'80)

Choice Set

(Restrict to "single-peaked" preferences)

Restricted Preference Domains

• Two-sided matching
  – Outcome?
  – All strict preferences possible?

• House allocation/Housing market
  – Outcome?
  – All strict preferences possible?

• Paired-Kidney Donation
  – Outcome?
  – All strict preferences possible?
I: Two-sided Matching

• Public school matching; Medical interns to residencies; TAs to professors

• Agents in two sets; one kind with strict preferences over other kind

• A matching: each agent assigned to at most one agent on other side

Example
An unstable matching

the boys

<table>
<thead>
<tr>
<th>Jake</th>
<th>Heart</th>
<th>Holly &gt; Claire &gt; Twiggy &gt; Jill</th>
</tr>
</thead>
</table>

and girls

<table>
<thead>
<tr>
<th>Twiggy</th>
<th>Heart</th>
<th>Jake &gt; Elwood &gt; Curtis &gt; Ray</th>
</tr>
</thead>
</table>

Elwood

<table>
<thead>
<tr>
<th>Claire &gt; Jill &gt; Twiggy &gt; Holly</th>
</tr>
</thead>
</table>

Curtis

<table>
<thead>
<tr>
<th>Twiggy &gt; Jill &gt; Holly &gt; Claire</th>
</tr>
</thead>
</table>

Ray

<table>
<thead>
<tr>
<th>Holly &gt; Claire &gt; Twiggy &gt; Jill</th>
</tr>
</thead>
</table>

Stable matching

- A stable matching is a matching with no blocking pair
- A blocking pair = two agents who prefer each other to match
Some questions

• Do stable matchings exist?
• Are they easy to find?
• Does stability matter?
• Are stable matchings unique?
• What about incentives?

Two-sided matching

• n agents on each side
• strict preference orders
• find a matching
  – each agent matched to at most one the other side
• (Generally: can “truncate” preference order, remain unmatched)
• Concerns:
  – incentive compatibility
  – stability
Boy-Proposing Deferred Acceptance

(Gale-Shapley 1962)

Termination

Fact 1.
Deferred acceptance terminates.

Why:
1. In each step, someone removes a name from their list.
2. Only a finite number of names, so cannot run forever.
Everyone matches (without preference truncation)

Fact 2.

Why:

Assume for contradiction: boy b isn’t matched.
1. Then b has crossed everyone off his list.
2. So each girl holds an offer at some point.
3. But, once a girl holds an offer, always holds.
4. So, all girls match
5. Number of girls equals number of boys.
6. So, b must be matched.

CONTRADICTION!

(Immorlica/Hartline)

Stable matchings exist

• Thm: Boy-proposing DA generates a stable matching

Proof.
• Suppose (b,g), (b’,g’) in the matching
• For contradiction, suppose (b,g’) blocking pair
• => blocking pair, so boy b prefers g’ to g, and proposes to g’ before girl g
• => girl g’ matches b’, and so prefers b’ to b (since match improves for girls during the DA)
• => (b,g’) is not a blocking pair. Contradiction.
Some questions

- Do stable matches exist? yes
- Are they easy to find? yes
- Does stability matter?
- Are stable matches unique?
- What about incentives?

<table>
<thead>
<tr>
<th>Market</th>
<th>Stable</th>
<th>Still in use</th>
</tr>
</thead>
<tbody>
<tr>
<td>NRMP</td>
<td>yes</td>
<td>yes (redesign '98)</td>
</tr>
<tr>
<td>Edinburgh ('69)</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Cardiff</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Birmingham</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Edinburgh ('67)</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Newcastle</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Sheffield</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Cambridge</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>London Hospital</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Medical Specialties</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Canadian Lawyers</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Dental Residencies</td>
<td>yes</td>
<td>yes (5 of 7)</td>
</tr>
<tr>
<td>Osteopaths (&lt; '94)</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Osteopaths (≥ '94)</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Pharmacists</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Reform rabbis</td>
<td>yes (first used in '97-98)</td>
<td>yes</td>
</tr>
<tr>
<td>Clinical psych</td>
<td>yes (first used in '99)</td>
<td>yes</td>
</tr>
</tbody>
</table>

Stability looks like an important feature of a centralized labor market clearinghouse. (Roth/Peranson)
Some questions

- Do stable matches exist? yes
- Are they easy to find? yes
- Does stability matter? yes
- Are stable matches unique?
- What about incentives?

Girl-Proposing Deferred Acceptance

Round 1
- Jake
  - Holly > Claire > Twiggy > Jill
- Elwood
  - Claire > Jill > Twiggy > Holly
- Curtis
  - Twiggy > Jill > Holly > Claire
- Ray
  - Holly > Claire > Twiggy > Jill

Round 2
- Twiggy
  - Jake > Elwood > Curtis > Ray
- Claire
  - Jake > Curtis > Elwood > Ray
- Jill
  - Ray > Curtis > Elwood > Jake
- Holly
  - Ray > Jake > Elwood > Curtis

Stop! (Stable)
Some questions

• Do stable matches exist? yes
• Are they easy to find? Yes
• Does stability matter? Yes
• Are they unique? No
• What about incentives?
Achievable outcomes

In general, there are many stable matchings.

\[ \text{Boy } b \quad \text{Girl } g \quad \text{Girl } g' \]

"achievable"

girl \( g \) is achievable for \( b \) if \( b \) and \( g \) match in some stable matching

\( \{g, g'\} \) are achievable for \( b \)

Given truthful reports

In boy-proposing DA:
1. Every boy matches with his most preferred, achievable girl
2. Every girl matches with her least preferred, achievable boy

*vice versa* for girl-proposing DA.
Each boy to preferred, achiev. girl

Truthful reports.
Assume for contradiction: some boy isn’t matched to favorite achievable girl
1. Let b be 1st boy to lose his favorite achievable g (and so b-g in a stable matching $\mu$)
2. Exists some b’ for which g prefers b’ to b.
3. Since b’ has not crossed off his favorite achievable girl, b’ prefers g over all achievable girls
4. Contradiction: {b-g , b’-g’) (for some g’) in stable $\mu$, but (b’,g) is a blocking pair!

(Immorlica/Hartline)

But of course … symmetry

If girls propose, then they will each match with their most preferred achievable boy

(Immorlica/Hartline)
Matchings have a lattice structure

- Given stable \( \mu_1 \), stable \( \mu_2 \)
- “join” (for boys): construct stable \( \mu' \) that is best for each boy simultaneously
- “meet” (for boys): construct stable \( \mu' \) that is worse for each boy simultaneously

Incentives

- Is truthful reporting a dominant strategy for boys in boy-proposing DA?
  - Yes. If truthful, boy \( b \) is matched to most-preferred, achievable girl (given reports of others); cannot do better.
- Girl-proposing DA is truthful for girls

- Attain an outcome that is stable with respect to reported preferences
- Negative result: No matching mechanism is stable and strategy-proof.
Some questions

• Do stable matches exist? yes
• Are they easy to find? Yes
• Do stable matches matter? Yes
• Are they unique? No
• What about incentives? Yes, on one side.

Markets + Romance
Real-world Matching Markets

- Adoption of student-proposing NMRP in 1998
- Easier for students
- Practical concern: couples with preferences on pairs of positions

School Choice (Boston and New York):

- "Boston mechanism" was not stable or truthful
Real-world Matching Markets

**National Residency Matching Program (NRMP):**
- Adoption of student-proposing NMRP in 1998
- Easier for students
- Practical concern: couples with preferences on pairs of positions

**School Choice (Boston and New York):**
- “Boston mechanism” was not stable or truthful
- Fix: adopt student-proposing DA
- Easier, more fair, and allow for policy advice
- Practical concern: priorities for schools (siblings, walk zones)
II. One-sided Matching

- “House allocation” problem
  - No initial property rights
- “Housing market” problem
  - Initial property rights

- Agents have strict preferences on items
- Items don’t have preferences on agents 😊
House allocation

- Set of items
- Each agent has strict preferences on items
- Want truthful, Pareto optimal mechanism
- Solution?
- How make this fair?

House allocation

- Set of items
- Each agent has strict preferences on items
- Want truthful, Pareto optimal mechanism
- Solution? Serial dictatorship (SD)
- How make this more fair? Random SD
House allocation

- Set of items
- Each agent has strict preferences on items
- Want strategy-proof, **Pareto optimal** mechanism
- Solution? **Serial dictatorship (SD)**
- How make this more fair? **Random SD**

Conjectured to be only *anonymous*, Pareto optimal, truthful mechanism.

RSD is not ex ante Pareto optimal

- Ex post PO: no assignment that makes >=1 better off, rest just as happy
- Ex ante PO: no distribution that makes >=1 better off, rest just as happy
Next class

• Housing markets (1-sided matching with initial property rights)
• Paired-kidney exchange

• Later: Ethics on Wed 11/8 (Kate Vredenburgh, Philosophy)