1. [14 Points] Bidding Languages

(a) [3 Points] Prove that the OR language is expressive for superadditive valuations and only superadditive valuations.

(b) [2 Points] The XOR-of-OR language allows a bidder to submit a bid such as \( (AB, 4) \lor (CD, 6) \oplus (EF, 10) \lor (FG, 12) \). How would this bid be represented in the OR* language?

(c) [4 Points] How many dummy items are needed to represent a general XOR-of-OR bid as an OR* bid? Assume \( k \) OR clauses, each with \( \ell \) atoms.

(d) [1 Points] Is the XOR-of-OR language fully expressive? Why or why not?

(e) [2 Points] Give an informal argument for why OR* bids “look just like” OR bids from the perspective of winner determination once dummy items are introduced into the supply.

(f) [extra credit] A bidder with the majority valuation function values any package of size at least \( m/2 \) items at \$1, and any smaller package at \$0. Show that the majority valuation function requires \( \binom{m}{m/2} \) atomic bids to represent in the OR* language. [Hint: it is helpful to argue that you do not need atoms smaller than \( m/2 \).]

2. [18 Points] Winner determination

(a) [3 Points] Consider two bidders and three items, and XOR bids “\( (A, 1) \oplus (BC, 2) \)” from bidder 1 and “\( (AB, 2) \oplus (C, 3) \)” from bidder 2. Cyclic structure (Section 11.3.3) only applies to bids in the OR or OR* language (treating dummy items as items for the purpose of the structure). Explain why these bids cannot satisfy the cyclic structure property (Section 11.3.3) when expressed in the OR* language.

(b) [2 Points] Use one of the properties S1–S4 to explain why the winner determination problem for XS (XOR-of-Singletons) is tractable.
Dynamic programming can be used to solve winner determination with OR bids. Let \( W(S) \) denote the value of the optimal allocation that only allocates items in set \( S \subseteq G \). Dynamic programming computes \( W(S) \) for all \( |S| = 1 \), then \( W(S) \) for all \( |S| = 2 \), size 3, and so forth.

(c) [3 Points] For some set \( S \subseteq G \), provide an expression for \( W(S) \) as a function of atomic bids on package \( S \) and \( W(S') \) for sets \( S' \subset S \).

(d) [4 Points] Walk through the steps of dynamic programming on the winner determination problem in Figure 11.1 (assuming truthful bids). What record-keeping allows the optimal solution to be determined at the end of the algorithm?

(e) [3 Points] Why is the run-time of dynamic programming \( O(n2^m + 2^m) \) for \( n \) bidders and \( m \) items? When is this polynomial in the size of the input?

(f) [3 Points] What is the problem with the OR language? What goes wrong with this dynamic programming scheme if the bidding language is XOR rather than OR (give a simple example)? What bidding language can be adopted in place of OR that is both general and can be combined with this dynamic programming scheme?

3. [13 Points] VCG auction, Core-selecting auctions

(a) [3 Points] Prove that if (11.21) is violated for some set \( L \), there is an allocation to the bidders in \( L \) where the value created could be divided between the seller and bidders in \( L \) to make them all strictly better off.

(b) [2 Points] Use (11.22) to confirm, by setting payments equal to value (and thus, bidders’ payoffs to zero), that the core of the CA is always non-empty.

Consider an instance with four goods \( \{A, B, C, D\} \), and three bidders, with single-minded valuations \((A, 10), (ABCD, 19), (B, 8)\) respectively.

(c) [3 Points] What is the outcome of the VCG mechanism (with truthful bidding)? How can the losers collude and win and pay zero?

(d) [3 Points] A core outcome of an auction has the following properties, all defined with respect to reported valuations:

- outcome \( X \) is efficient
- no allocated bidder pays more than its value
- unallocated bidders pay zero
- Eq. (11.24) holds for all subsets of allocated bidders (including the set of all allocated bidders)

A core outcome is bidder-optimal if, in addition, the outcome satisfies the property of Defn. 11.8. Assuming truthful bidding, describe the bidder-optimal core outcome or set of bidder-optimal core outcomes on this input. Is the VCG outcome in the core?

(e) [2 Points] Now consider the deviation in which the losing bidders collude as in part (c). Describe the bidder-optimal core outcome or set of bidder-optimal core outcomes on this input. What do you notice?