1. Sponsored Search Auctions

(a) Consider a GSP auction with two positions, three bidders, and position effect 0.3 and 0.2 in positions 1 and 2, respectively (assume quality effect 1.)

Provide an example of a valuation profile for which truthful bidding is not a Nash equilibrium. Justify your answer.

Answer:

Slots are allocated in order of decreasing order of bids, and payment is the next highest bid.

Consider value-per click of $v_1 = 30$, $v_2 = 20$ and $v_3 = 10$. Assume bidders 2 and 3 bid truthfully. Now, if bidder 1 bids her true value, she would be allocated the top position, pay 20 per click, and have expected utility $0.3(30 - 20) = 3$. She can do better with a bid of 11, getting position 2 and for 10 per click, and with expected utility $0.2(30 - 10) = 4 > 3$.

(b) What are the per-click VCG payments in your example in part (a)? Compare them to the GSP prices.

Answer: Bidders 1 and 2 would receive positions 1 and 2, respectively. The expected payments would be:

$t_{vcg,1}(b) = (p_1 - p_2)v_2 + (p_2)v_3 = 0.1(20) + 0.2(10) = 3$

$t_{vcg,2}(b) = p_2v_3 = 0.2(10) = 2$

Converting to per-click prices (dividing by CTRs), we have:

$PPC_{vcg,1}(b) = t_{vcg,1}/0.3 = 3/0.3 = 10$

$PPC_{vcg,2}(b) = t_{vcg,2}/0.2 = 2/0.2 = 10$

Compared with GSP, the per-click price is lower for bidder 1 and the same for bidder 2.

(c) Give three reasons why GSP rather than VCG is generally used for sponsored search.

Possible answers:

- Revenue effect: the revenue of GSP is higher than that of VCG for the same bids, thus switching from GSP to VCG would lead to a short term fall in revenue.
- GSP is easier to describe
- Engineering cost to make the change from GSP to VCG
- Equivalence balanced bidding outcome of GSP and truthful bidding outcome of VCG
2. Combinatorial auctions

(a) A bidder wants to win a New York area wireless spectrum license and a Boston area wireless spectrum license. Briefly, what could go wrong in an auction in which must bid on each item separately, and how is this addressed in a combinatorial auction?

Answer:
The problem is one of financial exposure. Without package bids, she might win only one of the two licenses and if her value is superadditive she might have bid more than her value for the item by itself.

In a combinatorial auction, a bidder can bid directly on the package of interest, and does not run this risk of receiving one item without the other item that makes the first item valuable.

(b) From the perspective of winner determination, why can we assume OR bids?

Answer: A winner determination algorithm that can find an optimal solution with an input represented in the OR language can also find an optimal solution for an input represented in the OR* language. This is because OR* bids are OR bids with additional “dummy items” to allocate.

(c) Express the valuation “I have an additive value for different cars, and want at most three cars” as a succinct OR-of-XOR bid or XOR-of-OR bid. (No need to give the full expression).

Answer:

\[
((C_1,\$500) \oplus (C_2,\$1000) \oplus (C_3,\$2000) \oplus \ldots \oplus (C_{100},\$1500)) \\
\lor ((C_1,\$500) \oplus (C_2,\$1000) \oplus (C_3,\$2000) \oplus \ldots \oplus (C_{100},\$1500)) \\
\lor ((C_1,\$500) \oplus (C_2,\$1000) \oplus (C_3,\$2000) \oplus \ldots \oplus (C_{100},\$1500))
\]

(d) Express the valuation “I’m only interested in particular pairs of bikes, but I’d take any number of the pairs I’m interested in” as a succinct OR or XOR bid. (Just show how you’d do this by illustrating this with a few pairs.)

Answer:

\[
\{B_1, B_3\}, \$200) \lor (\{B_1, B_3\}, \$300) \lor (\{B_3, B_6\}, \$150) \lor (\{B_7, B_8\}, \$450)
\]

(e) Would you use the OR or XOR logical operator to combine your answers to (c) and (d) and express the valuation for someone interested in cars and bikes, but not a mixture?

Answer:
The XOR operator. The overall bid structure would be “(bids for cars) \oplus (bids for bikes)”.
3. Paired-kidney exchange

Suppose a patient can receive a kidney if the patient’s blood includes all proteins in the blood of the donor (e.g., an A-type can receive O and A, but not from B or AB).

(a) There are four patient-donor pairs, with (patient-donor) blood types \{B-A, AB-B, AB-A, A-A\}. Draw an undirected graph to indicate the feasible swaps.

Answer:

\( B-A \implies AB-B \implies AB-A \implies A-A \)

(b) What is the maximum cardinality matching on this input (for swaps)?

Answer:

\( B-A \) with \( AB-B \), and \( AB-A \) with \( A-A \).

(c) How would the graph representation be modified in order to allow for matches that use 3-cycles, in addition to swaps?

Answer:

It becomes a directed graph, with an edge pointing from one pair to another if the donor in the first pair is compatible with the recipient in the second pair.

(d) How can altruistic donors be represented in this graph, and what impact do they have beyond just providing an additional transplant opportunity?

Answer:

We can introduce a new vertex for each donor, with a directed edge from the vertex to any patient-donor pair where the altruistic donor is blood-type compatible with the patient. We can also introduce a zero weight, “back edge” from every patient-donor pair back to the altruistic donor.

Additional impact: they allow for long chains of donations, since we can break up the transplants and do them one at a time, free from the ethical concern of leaving a pair without its donor.

(e) What is a new computational challenge when matching with altruistic donors?

Answer: Now that we’re able to accept long chains, it becomes very computationally expensive to find an optimal matching because there can be a very large number of possible chains.
4. Top-trading cycle mechanism

(a) Consider seven agents, with preference orders:

1: 2 \succ_1 3 \succ_1 1
2: 3 \succ_2 4 \succ_2 5 \succ_2 2
3: 1 \succ_3 3
4: 2 \succ_4 1 \succ_4 5 \succ_4 4
5: 4 \succ_5 5
6: 5 \succ_6 7 \succ_6 6
7: 5 \succ_7 6 \succ_7 7

The preference orders are truncated at the point where an agent prefers the house that she already owns. What trades occur in each round of the mechanism?

Answer:

round 1: 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7, trade is 1-2-3, so assignment is (1,H2), (2,H3), (3,H1).

round 2: now 4 \rightarrow 5 and 5 \rightarrow 4, 6 \rightarrow 5, 7 \rightarrow 5,. trade is on the 4-5 cycle, and assignment is (4,H5) and (5,H4)

round 3: now 6 and 7 point to each other and trade, and assignment is (6,H7) and (7,H6)

(b) Explain why the assignment produced by the TTC mechanism is Pareto Optimal. You can argue with respect to this particular example if you choose.

Answer:

Suppose for contradiction that there is an assignment \( x' \), distinct from the TTC assignment, that Pareto dominates the TTC assignment. Certainly it cannot disagree with the one we computed for agents 1, 2 or 3. But then it cannot disagree for agents 4 and 5 because they both get their most preferred item of those remaining. Likewise for agents 6 and 7. Thus, \( x' \) is the same as the TTC assignment. A contradiction!

(c) Suppose the random serial dictator (RSD) mechanism was used instead, where the houses of the seven agents were first placed into a common ownership pool. What property of TTC is not achieved by RSD?

Answer:

Either the “core” property (coalitions can do better by trading amongst themselves), or, as a special case, it is not individual rational and someone may end up with a worse house than the one she started with!

(d) State two differences between the 3-cycle matching problem in paired-kidney exchange and the matching problem solved in a single round of the TTC mechanism.

Answer:

variations on these are also ok

(a) KX restricts cycles to length 2 or 3, can be any length in TTC
(b) Cycles in TTC can be identified in linear time by walking from every vertex and looking for a cycle. The KX matching problem is NP-hard (for bounded cycle lengths greater than 2).
5. Two-sided Matching

Consider the following preference orders for four firms and four workers (most preferred to least preferred):

<table>
<thead>
<tr>
<th></th>
<th>f1: w1 w2 w3 w4</th>
<th>w1: f4 f3 f2 f1</th>
</tr>
</thead>
<tbody>
<tr>
<td>f2: w2 w1 w4 w3</td>
<td>w2: f3 f4 f1 f2</td>
<td></td>
</tr>
<tr>
<td>f3: w3 w4 w1 w2</td>
<td>w3: f2 f1 f4 f3</td>
<td></td>
</tr>
<tr>
<td>f4: w4 w3 w2 w1</td>
<td>w4: f1 f2 f3 f4</td>
<td></td>
</tr>
</tbody>
</table>

(a) Consider the following two stable matchings

matching 1: (f1, w2) (f2, w1) (f3, w3) (f4, w4)
matching 2: (f1, w1) (f2, w2) (f3, w4) (f4, w3)

What is the “join” of these two matchings (where each firm gets its most-preferred worker across both matchings)?

Answer:
join: (f1, w1) (f2, w2), (f3, w3), (f4, w4)

(b) What do you notice about this matching?

Answer:
every firm receives its most preferred worker, and thus the matching is stable.

Consider the following preference orders for three firms and three workers (most preferred to least preferred):

<table>
<thead>
<tr>
<th></th>
<th>f1: w1 w2 w3</th>
<th>w1: f1 f2 f3</th>
</tr>
</thead>
<tbody>
<tr>
<td>f2: w1 w3 w2</td>
<td>w2: f1 f2 f3</td>
<td></td>
</tr>
<tr>
<td>f3: w1 w2 w3</td>
<td>w3: f1 f3 f2</td>
<td></td>
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</tbody>
</table>

(c) There’s something about the firms’ preferences that seems to make this matching instance difficult. What is it?

Answer:
they all prefer worker 1. It is not clear how to decide who should get the worker.

(d) How does stability address this concern in general, and in this example in particular?

Answer:
in general, by considering preferences on both sides of the market.
here, we need (f1, w1) for stability, otherwise this will be a blocking pair. By contrast, (f2, w1) and (f3, w1) are not blocking pairs. So, stability breaks this apparent tie in favor of f1.
To see this, we can also run firm-proposing DA:
round 1: (f1, w1)
round 2: (f2, w2) (f3, w3)
and worker-proposing DA
round 1: (f1, w1)
round 2: (f2, f3) (f3, w2)
6. Peer prediction

(a) Consider a setting with two possible signals $\ell$ and $h$. Suppose the output-agreement mechanism is used, paying $1$ for agreement and $0$ otherwise. What needs to be true of $\Pr(X_2 = h \mid X_1 = h)$ and $\Pr(X_2 = \ell \mid X_1 = \ell)$ for the mechanism to be strictly proper? **Provide a brief justification.**

**Answer:**

We need $\Pr(X_2 = h \mid X_1 = h) > \Pr(X_2 = \ell \mid X_1 = h)$ and $\Pr(X_2 = \ell \mid X_1 = \ell) > \Pr(X_2 = h \mid X_1 = \ell)$. If this property holds, then the expected payment for truthful reporting is strictly greater than a misreport. For example,

$$E(\text{score 1} \mid \text{report } h, \text{signal } h) = 1 \times \Pr(X_2 = h \mid X_1 = h) > 1 \times \Pr(X_2 = \ell \mid X_1 = h)$$

and similarly if the received signal is $\ell$.

(b) Suppose the distribution on signals by agents 1 and 2 is

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th>Player 1</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$h$</td>
<td>$\ell$</td>
</tr>
<tr>
<td>$h$</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>$\ell$</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

What is the design of the $1/prior$ mechanism in this case? Confirm that the mechanism is strictly proper.

**Answer:**

Because the marginal probability $P(h) = P(h, h) + P(h, \ell) = 0.5 + 0.2 = 7/10$ and $P(\ell) = P(\ell, h) + P(\ell, \ell) = 0.2 + 0.1 = 3/10$, the payments made for different reports are:

<table>
<thead>
<tr>
<th></th>
<th>$h$</th>
<th>$\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$10/7, 10/7$</td>
<td>0, 0</td>
</tr>
<tr>
<td>$\ell$</td>
<td>0, 0</td>
<td>$10/3, 10/3$</td>
</tr>
</tbody>
</table>

Suppose the signal received by agent 1 is $h$. Then, for strictly proper we need

$$(5/7)(10/7) > (2/7)(10/3) \iff 50/49 > 20/21,$$

so that the expected payment is greater for report $h$ than report $\ell$. This is true. Suppose the signal received by agent 1 is $\ell$. Then, we need

$$(1/3)(10/3) > (2/3)(10/7) \iff 10/9 > 20/21,$$

so that the expected payment is greater for report $\ell$ than report $\ell$. This is also true.

(c) Suppose there are three possible signals $\{1, 2, 3\}$. Your friend has designed a peer prediction method that satisfies:

$$E_{s_2 \sim P(X_2 \mid X_1 = s_1)}[t_1(s_1, s_2)] > E_{s_2 \sim P(X_2 \mid X_1 = r_1)}[t_1(r_1, s_2)], \quad \forall s_1 \in \{1, 2\}, \forall r_1 \neq s_1.$$
It does not satisfy this property for $s_1 = 3$. Likewise, the same inequality holds for agent 2 and signals $s_2 \in \{1, 2\}$ (but not $s_2 = 3$).

Will it be a strict best response for player 1 to report her signal truthfully when $s_1 \in \{1, 2\}$? Justify your answer.

**Answer:**

Not necessarily, because there is no reason to expect the reports of player 2 for $s_2 = 3$ to be truthful. Because of this, the inequality is insufficient for strict properness because it assumes player 2 reports every signal truthfully, including signal 3 (when it takes the expectation with respect to $s_2 \sim P(X_2 | X_1 = s_1)$).
7. Miscellaneous

(a) Briefly, what two properties of the Blockchain design prevent anyone from easily printing Bitcoins and giving them to themselves?

Answer:
(1) Proof of work— your new block is only accepted onto the blockchain if it includes a proof-of-work, making it costly to add a new block onto the head of the chain.

(2) Rules of the game— given proof-of-work, you can still only print for yourself as many coins as currently allowed by the protocol, otherwise the new block will not be acceptable.

(b) Denoting a possible outcome \( o_k \) (one of \( k \in \{1, \ldots, m\} \)), write down the definition of strict properness for a scoring rule \( s(q, o) \) that takes belief \( q \) and realized outcome \( o \).

Answer:
Letting \( p \) denote the agent’s true belief, strict properness requires:

\[
\sum_{k=0}^{m-1} p_k \cdot s(p, o_k) > \sum_{k=0}^{m-1} p_k \cdot s(q, o_k), \quad \forall q \neq p
\]

(c) Briefly, what goes wrong with a linear scoring rule and how is this addressed with the logarithmic or quadratic scoring rules?

Answer:
The linear rule provides incentives for ‘extremal’ reports, e.g. reporting \( q_{\text{sun}} = 1 \) if I think it is more likely than not to be sunny. This is fixed in log and quadratic by providing a marginal-decreasing increase in value for reporting a belief closer to 1 in the event that this outcome occurs.

(d) Give an example of adverse selection, and use that example to explain how a well-functioning reputation system would address the problem of adverse selection.

Answer:
Low quality plumbers. Fix: by having a reputation system, a low quality plumber is soon identified as one and can no longer easily find business.

(e) State two aspects of the “reputation game” model and analysis that make it overly stylized relative to real-world reputation systems.

Answer:
(other answers are also valid, we give three here)

(1) any low quality outcome from a transaction makes its way into affecting your reputation (vs. a real system where feedback may be missing)

(2) reputation is binary, and even one low quality leads to a bad reputation

(3) as soon as reputation is bad then every other agent knows about this and refuses to cooperate