Total points: 49
This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of answers, and we encourage typed submissions. You are free to discuss the problem set with other students but you must not share your answers. Extra credit will only be considered as a factor in deciding the letter grade for the course at the end of the term. Submit to Canvas.

1. [12 Points] Two-sided matching
   (a) [2 Points] Verify that the matching selected by the teacher-proposing DA in Example 12.3 is (weakly) better for every teacher but (weakly) worse for every student than the matching selected by the student-proposing DA.
   (b) [4 Points] Prove that there exists no mechanism for two-sided matching that is both stable and strategyproof. [Hint: consider the following preference orders for $s_1, s_2, t_1, t_2$: $t_1 \succ s_1 t_2, t_2 \succ s_2 t_1, s_2 \succ t_1 s_1, \text{and } s_1 \succ t_2 s_2$. “Truncation” misreports are allowed.]
   (c) [2 Points] Complete the argument in the chapter (end of Section 12.2.2) to explain why the outcome of student-proposing DA can be computed in $O(mn)$ steps, for $m$ students and $n$ teachers.
   (d) [2 Points] Use the lattice property of stable matchings (end of Section 12.2.2) to prove that every student weakly-prefers any stable matching to the teacher-optimal matching.
   (e) [2 Points] Show that if the student-optimal and teacher-optimal stable matchings are the same then there is a unique stable matching.

2. [15 Points] Many-to-one matching
   Consider a many-to-one two-sided matching problem between students and research hospitals. Each hospital $h$ has $q_h \geq 1$ positions available, and a strict preference order on individual students and responsive preferences for sets of students. Responsive preferences means that, for any set $S$ of students with $|S| < q_h$, and any students $s$ and $s'$ not in $S$, hospital $h$ prefers $S \cup \{s\}$ to $S \cup \{s'\}$ if and only if student $s$ is preferred by $h$ to student $s'$. Moreover, $(S \cup \{s\}) \succ_h S$ if and only if $s \succ_h \phi$ (denoting that $s$ is ‘acceptable’).
   (a) [5 Points] Describe a natural way to extend DA to many-to-one matching, considering both student-proposing and hospital-proposing. Explain the role of responsive preferences in your definition.
   (b) [4 Points] There are two hospitals $h_1, h_2$, quotas $q_{h_1} = 2, q_{h_2} = 1$, and two students $s_1$ and $s_2$. The preference orders are $h_1 \succ s_1 h_2, h_2 \succ s_2 h_1, s_2 \succ h_1 s_1 \text{and } s_1 \succ h_2 s_2$. Verify that student-proposing and hospital-proposing DA produce the same matching on this input, and check that the matching is stable.
(c) [1 Points] State three properties that you can conclude about this stable matching? [Hint: the lattice structure of stable matchings continues to hold for many-to-one matching.]

(d) [4 Points] Consider a misreport by $h_1$ to $s_2 \succ_{h_1} \emptyset$, so that the hospital claims that only student $s_2$ is acceptable. Use this to prove that there exists no mechanism for many-to-one two-sided matching that is stable, and also truthful for hospitals. [Hint: run student-proposing and hospital-proposing DA with this perturbed preference profile.]

(e) [1 Points] Briefly, what useful policy advantage comes from student-proposing being truthful for students?

3. [11 Points] Public school choice

Suppose for simplicity that each school has capacity one. Consider the following preference orders for students $\{a_1, a_2, a_3\}$, and priority orders for schools $\{b_1, b_2, b_3\}$. The priorities of schools may reflect a preference for under-represented minorities, or walk-zone students.

$\succ_{a_1}: b_2 b_1 b_3 \succ_{b_1}: a_1 a_3 a_2$
$\succ_{a_2}: b_1 b_2 b_3 \succ_{b_2}: a_2 a_1 a_3$
$\succ_{a_3}: b_1 b_2 b_3 \succ_{b_3}: a_2 a_1 a_3$

(a) [3 Points] Use student-proposing DA to find the student-optimal, stable matching. Is there a matching (perhaps unstable) that Pareto dominates this matching for students?

(b) [4 Points] The Boston mechanism (used in Boston high schools until 2005) is defined as follows:

In step one, each student proposes to his or her first choice school, and students are matched with a school in order of school priority while there remains capacity.

In each subsequent step $k > 1$: each unmatched student proposes to his or her $k$th most-preferred school, and students are matched with a school in order of school priority while there remains capacity. The mechanism terminates when all students are matched.

Run the Boston mechanism on the example in part (a). Show that student $a_2$ has a useful misreport. Give a general description of the kind of manipulation that can be useful in the Boston mechanism.

(c) [4 Points] A variation on the top-trading-cycles mechanism can be applied to public school choice:

In each step, each school with remaining capacity points to the unmatched student with highest priority, and each unmatched student points to his or her most-preferred school with remaining capacity. Paths alternate between students and schools, and “trading on a cycle” corresponds to each student on the cycle being matched with his or her requested school.

Run this mechanism on the example. Compare the stability and Pareto-optimality (for students) with the matching obtained by student-proposing DA. Can you interpret the mechanism as students trading priorities amongst themselves? By analogy to TTC, do you think the mechanism is strategy-proof for students (no need for a proof)?
4. **[11 Points]** Paired Kidney Donation

(a) **[4 Points]** The following is an “edge formulation” of the problem of finding a maximum cardinality, vertex-disjoint matching. Interpret the variables, objective and constraints. Does the formulation limit cycle lengths? In terms of the number of pairs and compatibilities, how many decision variables and constraints are there in the edge formulation?

$$\max \sum_{(u,v) \in E} y_{uv}$$

s.t. \[ \sum_{v \in V} y_{uv} \leq 1, \quad \forall u \in V \] \quad (1)

\[ \sum_{v \in V} y_{uv} = \sum_{v \in V} y_{vu}, \quad \forall u \in V \] \quad (2)

\[ y_{uv} \in \{0, 1\}, \quad \forall u, v \in V \]

(b) **[2 Points]** How does the size of the edge formulation compare to the size of the cycle formulation for cycles of length up to three (see top p. 310).

(c) **[1 Points]** What is a practical problem with the edge formulation when using the matching for kidney exchange?

(d) **[4 Points]** Figure 12.5 (c) shows an example of how an AB-O pair can be used on a 3-cycle to enable two pairs (O-A and A-AB) to match that would be unable to match otherwise. Provide examples of feasible 3-cycles that can form with each of the following pairs:

(i) AB-O (give one additional example)
(ii) AB-A (give one example)
(iii) B-O (give two examples)

extra credit Give an example of a 4-cycle that includes pair AB-O, and in which 3 pairs can match that would not otherwise be able to match.