1 Mechanism Design

1.1 Review

Key Points to Understand

1. Single Parameter Domain: each agent has a private input $w_i \in \mathbb{R}$, and a summarization function $q_i : A \rightarrow \mathbb{R}_{\geq 0}$ (known to the mechanism designer). For an allocation $a$, agent $i$’s valuation function is $v_i(w_i, a) = w_i \cdot q_i(a)$. Examples include single item (possibly multi-unit) auction. A mechanism $(x, t)$ is strategy-proof (SP) for a single parameter domain if and only if, for all $i$ and $w_i$,

- (monotonicity) A choice rule $x$ is **monotone non-decreasing**, if, for all $i \in N$, all $\hat{w}_i$, $q_i(x(\hat{w}_i, \hat{w}_-)) \geq q_i(x(w_i, \hat{w}_-))$ for all $w_i' > w_i$.
- (payment identity) payment rule $t$ satisfies

$$t_i(\hat{w}_i, \hat{w}_-) = \hat{w}_i \cdot q_i(x(\hat{w}_i, \hat{w}_-)) - \int_{z=0}^{\hat{w}_i} q_i(x(z, \hat{w}_-))dz \quad (1)$$

2. Knapsack: we have some number $m \geq 1$ of identical, indivisible items to sell to a set of $n$ bidders. Each bidder $i$ demands a public quantity $k_i$ of items, but her private value $w_i$ on her bid is private; so, we have a report $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_n)$ of items. The **knapsack auction** is defined by the following allocation and payment rules:

- The allocation rule is as follows: first let $W_g$ be the value of the greedy allocation $N_{approx}$: remember that this ranks bidders in decreasing order of bang-for-buck, $\hat{w}_i/k_i$, and allocates in decreasing order, skipping a bid if it exceeds capacity. Let $W_h = \max_i \hat{w}_i$ be the value of the largest bid (assuming all $k_i$ are less than capacity). If $W_g \geq W_h$ items are allocated according to $N_{approx}$, else the highest bidder wins.
- The payment rule: simply the critical value (from single-parameter domains). In other words, 0 if a bidder is unallocated, else the minimum amount she must bid to be allocated.

3. Single-minded CA: there are bidders $N = \{1, \ldots, n\}$, a set $G$ of distinct and indivisible items, bidders report pairs $(\hat{T}_i, \hat{w}_i)$ (with true valuations $(T_i, w_i)$), with $T_i, \hat{T}_i \subseteq G$, and $w_i, \hat{w}_i \geq 0$. If bidder $i$ is allocated set $S$, then $v_i(S)$ is 0 if $T_i$ is not contained in $S$, else $w_i$. Suppose that $\sigma : 2^G \times \mathbb{R} \rightarrow \mathbb{R}$ is a monotone scoring function (so that $\sigma(w_i', T_i') \geq \sigma(w_i, T_i)$, if either $w_i' \geq w_i$ or $T_i' \subseteq T_i$ (or both)). The single-minded CA is defined by:
• Allocation rule: sort bids in order of decreasing score; accept greedily, skipping a bid if one or more items in the set $\hat{T}_i$ has been allocated.

• Payment: bidder $i$ is allocated by its critical value.

4. Min-makespan scheduling: $G$ is a set of tasks, $|G| = m$, and $c_{ij} > 0$ is the time agent $i \in N$ takes to complete task $j \in G$. Let $z \in Z$ denote a feasible assignment of tasks. The design objective is to minimize $\text{makespan}(z)$ over all assignments $z$, defined by

$$\text{makespan}(z) = \max_{i \in N} \left[ \sum_{j \in z_i} c_{ij} \right].$$

Theorem: for 2 agents, no deterministic mechanism can achieve an approximation ratio of better than 2 for the general problem of min-makespan scheduling.

So, we focus on the special case of a single-parameter setting: each agent has a type $w_i = -r_i$, the negative unit processing time. $v_i(w_i, z) = -r_i \cdot (\sum_{j \in z_i} l_j) = -r_i \cdot q_i(z)$, where $q_i(z)$ is the total amount of work provided to agent $i$. For the single-parameter problem, there is a deterministic SP optimal min-makespan mechanism (pset!).

5. Agent-independent pricing function: for every $v_{-i}$ there does not exist two different reports $v_i, v_i'$ of agent $i$ such that the same outcome is selected but $t_i(v_i, v_{-i}) \neq t_i(v_i', v_{-i})$. In other words, for all $v_{-i}$, the payment is independent of $v_i$ as long as the same outcome is chosen. This together with “agent optimizing” (agents get what they want when facing these prices) are necessary and sufficient for SP.

1.2 Practice

1. TV Advertising

Consider an auction where advertisers are bidding for Superbowl commercials. There are spots for three commercials to air and the advertiser with the first spot gets 40 views, the second gets 25 views, and the third gets 10 views (numbers in millions). Each advertiser wants one of the three commercials and only cares about the number of views she gets.

(a) Assume a mechanism designer runs an auction where bidders bid their willingness to pay per view. Come up with a choice rule that is monotone non-decreasing in every report $\hat{v}_i$.

(b) Consider the simple monotone non-decreasing choice and payment rules where the highest bidder receives the top spot and pays the second highest price per view; the second highest bidder receives the second spot and pays the third highest price per view, and the third highest bidder receives the third spot and pays the fourth highest price per view. Is this mechanism strategyproof? If so, show that it is strategyproof, if not come up with an example where it is not strategyproof.

(c) Come up with a payment mechanism that is strategyproof using the payment identity from the reading on mechanism design (eq. [1]). Suppose there are four advertisers participating in the auction and the bids of the first three agents are 4, 6, and 10. Draw the allocation vs value curve and the utility vs value curve for a fourth agent with valuation $0 \leq v_4 \leq 20$.

1Technically this domain is not single-parameter, but it is shown in the reading that the knapsack auction is SP with this payment rule.
2. **Binary domain** A sealed-bid auction for a single item (and values on 0,1) is defined in terms of an allocation rule \( x : [0,1]^n \rightarrow \{0,1\}^n \) and payment rule \( t : [0,1]^n \rightarrow \mathbb{R}^n \). For a given valuation profile \( v \), \( x_i(v) = 1 \) if agent \( i \) gets the item and it is 0 otherwise. Recall the notions of critical value payment (an allocated bidder is charged the minimum bid so that she is allocated the item) and monotone allocation (the choice rule \( x_i \) is monotone non-decreasing in an agent’s report \( \hat{v}_i \)).

(a) Prove that any auction that has a monotone allocation rule and charges critical value payments has a dominant-strategy equilibrium where agents report truthfully.

(b) Define the second-price Vickrey auction in these terms (i.e. show the choice rule is monotone and that bidders are charged their critical value.)

3. **VCG-based and single-minded CA** Using the below example, show that a VCG-based mechanism that uses the greedy algorithm for the single-minded CA is not strategy-proof. Show that all bidders have a useful manipulation. What do you notice about the effects of the manipulation on the allocation?

The example is: there are 3 items \( A, B, C \), and 3 bidders with target valuations \((AB, 3), (AC, 2)(B, 2)\). The value-based scoring function \( \sigma(T, \hat{w}_i) = \hat{w}_i \) is used.

4. **Knapsack auction** Recall that in the knapsack auction the allocation is made by the greedy allocation if \( W_g \geq W_h \), else the bidder with highest bid \( (W_h) \) is allocated.

(a) Find an example of an instance in which the total value by greedy allocation \( W_g \) is greater than the largest bid value \( W_h \). Check that no agent has a useful deviation.

(b) Find an example of an instance in which \( W_g < W_h \). Check that no agent has a useful deviation.

5. **Single-parameter min-makespan** Suppose there are 3 agents in a min-makespan scheduling problem, with unit-processing costs \( r_1 = 1, r_2 = 2, r_3 = 4 \), and 3 tasks \( l_A = 2, l_B = 3, l_C = 4 \). (a) What are the implied cost functions? (b) What is the min-makespan assignment? What is the minimum makespan?

6. **LexOpt** The LexOpt scheduling rule for the single-parameter makespan scheduling problem is defined as follows: if \( \hat{w} = (\hat{w}_1, \ldots, \hat{w}_n) \) is the report profile, then \( x(\hat{w}) \) selects a minimum-makespan assignment, breaking ties lexicographically: assignment \( z \) is preferred over \( z' \) if there exists an agent \( k \) such that \( q_k(z) < q_k(z') \) and \( q_i(z) = q_i(z') \) for all \( i < k \).

(a) For the example in the previous question, what is the allocation by LexOpt? If \( z \) denotes the allocation, then what is \( q_1(z), q_2(z), q_3(z) \)?

(b) Verify that if agent 1 reports \( \hat{r}_1 < 1 \), then the amount of work she is allocated \( q_1(z') \) will never decrease.

(c) What is the allocation of LexOpt if instead \( r_2 = 4.5 \) and \( r_3 = 8 \)?

### 1.3 More practice problems

1. (a) Take the buying-a-path-in-a-network problem, shown in the figure. Each edge corresponds to an agent \( i \) with cost \( c_i > 0 \) if the edge is used. Bob wants to get from S to F so he asks for
path costs of all the agents and runs the VCG mechanism. Which path does Bob purchase, and what are the VCG payments for the agents along the path?

(b) Bob uses the VCG mechanism to go from $S$ to $F$. But now wants to travel back home, from $F$ to $S$. He could use VCG again, but he can actually do better. What are some other path-procurement mechanisms that are better for Bob than using VCG? [Hint: now he knows more!]