1 Information Elicitation

The goal of effective information elicitation is to make it an equilibrium for participants to invest effort and report accurate information. The two main kinds of information elicitation tasks depend on whether the event being predicted can be verified (resolved) in future (e.g., will it rain tomorrow), or whether it’s something that’s not easily verifiable (e.g., how good a restaurant is for kids).

1.1 Review

1. Scoring rules attempt to incentivize agents to report their beliefs about some event/outcome that will be realized in the future.

A scoring rule $s$ takes a reported belief, and defines a payment to the forecaster, contingent on the outcome. Given a set of possible outcomes $O$, a scoring rule is a function $s : \Delta(O) \times O \rightarrow \mathbb{R} \cup \{-\infty\}$.

2. An agent’s payment (which can be negative) is a function of her report and the realized outcome. The expected payment for a forecaster with belief $p$ who reports belief $q$ (possibly $q \neq p$) is $S(q, p) = \mathbb{E}_{o \sim p}[s(q, o)] = \sum_{k=0}^{m-1} p_k \cdot s(q, o_k)$, where $o \sim p$ denotes that the outcome is distributed according to the probability $p$. The expectation is taken with respect to the forecaster’s true belief, and depends on the report $q$ and the realized outcome $o_k$.

3. A scoring rule is strictly proper if it incentivizes an agent to report her true belief. A scoring is strictly proper if, for every belief $p$, the expected payment is uniquely maximized through the truthful report.

4. The quadratic and logarithmic scoring rules are examples of strictly proper scoring rules.

5. The logarithmic scoring rule is $s_{\log}(q, o_k) = \ln(q_k)$.

6. The quadratic scoring rule (Brier’s rule) is $s_{\text{quad}}(q, o_k) = 2q_k - \sum_{k'=0}^{m-1} q_{k'}^2$.

7. If $s(q, o_k)$ is a strictly proper scoring rule on $m$ outcomes, then any rule $s'(q, o_k) = \alpha_k + \beta \cdot s(q, o_k)$ obtained via a positive affine transform with $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}_{>0}$ is strictly proper.

8. Sometimes agents must exert effort to figure out their beliefs. By increasing the value of $\beta$ in our scoring rule, we can make it worthwhile for agents to exert effort.
9. In the **peer prediction** method, agents are asked to report their signal and payments are made based on joint reports of the signals. There’s no verifiable ground truth against which to check reports. The agents report signals without knowing what the other agents’ reports are.

10. Two simple mechanisms are output agreement and 1/Prior. There are also mechanisms based on scoring rules.

a. Output Agreement

\[ t_i(r_1, r_2) = 1(r_1 = r_2) \text{ for } i \in \{1, 2\} \]

for reports \(r_1, r_2\) from agents 1 and 2, respectively. \(1 : X \rightarrow \{0, 1\}\), defined as 1 if \(x\) is true and 0 if \(x\) is false.

b. 1/Prior

\[ t_i(r_1, r_2) = \frac{1}{P(r_1)} 1(r_1 = r_2) \text{ for } i \in \{1, 2\} \]

for reports \(r_1, r_2\) from agents 1 and 2, respectively. \(P(r_1)\) specifies the probability that signal \(r_1\) arises in the distribution.

c. Scoring-Rule based Peer-Prediction. Let \(r_1\) and \(r_2\) denote the reports from agents of 1 and 2, respectively. The payment to agent 1 is determined by:

i. Compute a signal distribution \(q\), defined as the signal posterior distribution \(p(\ell | r_1)\) on the signal \(\ell\) of agent 2, given that agent 1’s signal is \(r_1\).

\[ q = Pr(X_2 | X_1 = r_1) \]

ii. Make payment \(t_1(r_1, r_2) = s(q, r_2)\) to agent 1, where \(s\) is any strictly proper scoring rule.

iii. The payment to agent 2 is determined analogously, using the signal posterior \(p(j | r_2)\) on the signal \(j\) of agent 1 given that agent 2’s signal is \(r_2\).

The scoring-rule based peer prediction mechanism is strictly proper if and only if the signal distribution satisfies stochastic relevance. A signal distribution satisfies stochastic relevance if and only if the signal-conditional distribution \(P(X_2 | X_1 = j)\) is distinct from the signal conditional distribution \(P(X_2 | X_1 = j')\), for any two signals.

d. Multi-task Bonus-Penalty Mechanism

\[ t_i(r_1, r_2) = 1(r_1^{(0)} = r_2^{(0)}) - 1(r_1^{(1)} = r_2^{(2)}) \text{ for } i \in \{1, 2\} \]

for reports \(r_1, r_2\) on a set of three tasks from agents 1 and 2, respectively. The superscripts \((0), (1), (2)\) are the indices of the three tasks.

11. One issue with running peer prediction mechanisms in practice is that there can be additional, uninformative equilibria. Participants in a peer-prediction mechanism may coordinate and receive high payments without revealing any useful information.

For example, the OA mechanism for Example 17.2 in the text has an uninformative equilibria. The action profiles \(r = (0, 0)\) and \(r = (1, 1)\) are Nash equilibria, and provide guaranteed
payment $1 to each agent. This is strictly greater than the expected payment from truthful
reports, which is $P(X_1 = 0, X_2 = 0) \cdot 1 + P(X_1 = 1, X_2 = 1) \cdot 1 = 0.4 + 0.4 = 0.8$
A good solution is to use the multi-task bonus/penalty mechanism (and its generalizations).

1.2 Exercises

1.2.1 Biased Coin

Suppose we have a biased coin that comes up heads with an unknown probability $p$. Based on our
observations of past coin flips, we form the belief that $p$ is $\frac{\text{number of } H}{\text{number of flips}}$.

We have flipped the coin 9 times and have received 3 heads.

1. Under a logarithmic scoring rule, what should I report as my belief about the probability that
the next flip is an H to maximize expected score?

2. What is my expected payoff from reporting truthfully?

3. What is my expected payoff from reporting 0.25?

4. What is my expected payoff from reporting 0.5?

Answer:

1. This scoring rule is strictly proper, so we want to report our best guess of the true value of $p$, namely $\frac{1}{3}$.

2. $E = \frac{1}{3} \cdot \ln\left(\frac{1}{3}\right) + \frac{2}{3} \cdot \ln\left(\frac{2}{3}\right) \approx -0.6365142$

3. $E = \frac{1}{3} \cdot \ln\left(\frac{1}{4}\right) + \frac{2}{3} \cdot \ln\left(\frac{3}{4}\right) \approx -0.6538862$

4. $E = \frac{1}{3} \cdot \ln\left(\frac{1}{2}\right) + \frac{2}{3} \cdot \ln\left(\frac{1}{2}\right) \approx -0.6931472$

1.2.2 Peer Prediction

Suppose we have true, hidden states of the world $H = \{0, 1\}$ and possible signals $S_i = \{0, 1\}, \forall i \in
\{1, 2\}$, and that $P(H = 0) = 0.4$. Further suppose that $P(X_i = 0|H = 0) = 0.6$ and $P(X_i = 1|H = 1) = 0.8$.

1. Calculate the joint distribution on signals.

Answer:

$$P(X_1 = 0, X_2 = 0) = P(H = 0, X_1 = 0, X_2 = 0) + P(H = 1, X_1 = 0, X_2 = 0)$$

$$= P(H = 0)P(X_1 = 0|H = 0)P(X_2 = 0|H = 0) + P(H = 1)P(X_1 = 0|H = 1)P(X_2 = 0|H = 1)$$

$$= 0.4(0.6)(0.6) + 0.6(0.2)(0.2) = 0.168.$$
\[
0.4(0.6)(0.4) + 0.6(0.2)(0.8) = 0.192
\]

\[
P(X_1 = 1, X_2 = 0) = P(H = 0, X_1 = 1, X_2 = 0) + P(H = 1, X_1 = 1, X_2 = 0) \\
= P(H = 0)P(X_1 = 1|H = 0)P(X_2 = 0|H = 0) + P(H = 1)P(X_1 = 1|H = 1)P(X_2 = 0|H = 1) \\
= 0.4(0.4)(0.6) + 0.6(0.8)(0.2) = 0.192.
\]

\[
P(X_1 = 1, X_2 = 1) = P(H = 0, X_1 = 1, X_2 = 1) + P(H = 1, X_1 = 1, X_2 = 1) \\
= P(H = 0)P(X_1 = 1|H = 0)P(X_2 = 1|H = 0) + P(H = 1)P(X_1 = 1|H = 1)P(X_2 = 1|H = 1) \\
= 0.4(0.4)(0.4) + 0.6(0.8)(0.8) = 0.448.
\]

2. Calculate the payments in truthful equilibrium under Output Agreement. Is OA strictly proper here? Justify your answer.

**Answer:**

\[
P(X_1 = 0, X_2 = 0) + P(X_1, X_2 = 1) = 0.168 + 0.448 = 0.616.
\]

OA is not strictly proper by Theorem 17.2 (the joint signal distribution is not self-dominant).

To see why, first calculate the marginal probabilities:

\[
P(X_1 = 0) = P(X_1 = 0, X_2 = 0) + P(X_1 = 0, X_2 = 1) = 0.168 + 0.192 = 0.36
\]

\[
P(X_1 = 1) = P(X_1 = 1, X_2 = 0) + P(X_1 = 1, X_2 = 1) = 0.192 + 0.448 = 0.64
\]

and symmetrically for agent \(X_2\).

Then,

\[
P(X_2 = 0|X_1 = 0) = \frac{P(X_2 = 0, X_1 = 0)}{P(X_1 = 0)} = \frac{0.168}{0.36} \approx 0.467
\]

\[
P(X_2 = 1|X_1 = 0) = \frac{P(X_2 = 1, X_1 = 0)}{P(X_1 = 0)} = \frac{0.192}{0.36} \approx 0.533
\]

\[
P(X_2 = 0|X_1 = 1) = \frac{P(X_2 = 0, X_1 = 1)}{P(X_1 = 1)} = \frac{0.192}{0.64} = 0.3
\]

\[
P(X_2 = 1|X_1 = 1) = \frac{P(X_2 = 1, X_1 = 1)}{P(X_1 = 1)} = \frac{0.448}{0.64} = 0.7
\]

Thus, \(P(X_2 = 1|X_1 = 0) > P(X_2 = 0|X_1 = 0)\) and so the joint signal distribution is not self-dominant.

3. Are there additional, uninformative equilibria under Output Agreement? If so, what are they?

**Answer:** Yes. Namely, (0,0) and (1,1) are PSNE. There is also a mixed-strategy Nash equilibrium with each agent reporting signal 0 with probability 0.5.
4. Now think about the scoring-rule based peer-prediction mechanism. Calculate payments under the logarithmic scoring rule for each pair of reports. Then, give transformed payments such that strict properness still holds but payments are between 0 and 1.

**Answer:**

Thus, payments are:

\[ r_1 = 0, r_2 = 0 : \ln(0.467) \approx -0.759287 \]
\[ r_1 = 0, r_2 = 1 : \ln(0.533) \approx -0.6292339 \]
\[ r_1 = 1, r_2 = 0 : \ln(0.3) \approx -1.203973 \]
\[ r_1 = 1, r_2 = 1 : \ln(0.7) \approx -0.3566749 \]

We can scale these payments so that they are between 0 and 1 for all reports. We want \( \alpha + \beta \cdot t_{\log(\bullet)} \) to be between 0 and 1.

One way to do this is to let \( \beta = \frac{1}{2} \) to get all payments to be between -1 and 0:

\[ r_1 = 0, r_2 = 0 : 0.5 \cdot \ln(0.467) \approx -0.380713 \]
\[ r_1 = 0, r_2 = 1 : 0.5 \cdot \ln(0.533) \approx -0.3146169 \]
\[ r_1 = 1, r_2 = 0 : 0.5 \cdot \ln(0.3) \approx -0.6019864 \]
\[ r_1 = 1, r_2 = 1 : 0.5 \cdot \ln(0.7) \approx -0.1783375 \]

Now, just let \( \alpha = 1 \). Then, we have

\[ r_1 = 0, r_2 = 0 : 1 + 0.5 \cdot \ln(0.467) \approx 0.619287 \]
\[ r_1 = 0, r_2 = 1 : 1 + 0.5 \cdot \ln(0.533) \approx 0.6853831 \]
\[ r_1 = 1, r_2 = 0 : 1 + 0.5 \cdot \ln(0.3) \approx 0.3980136 \]
\[ r_1 = 1, r_2 = 1 : 1 + 0.5 \cdot \ln(0.7) \approx 0.8216625 \]

5. Calculate payments under the 1/Prior peer prediction mechanism. Is this mechanism strictly proper? Justify your answer.

**Answer:** Pay \( \frac{1}{0.36} = 2.778 \) under \( r_1 = 0, r_2 = 0 \), pay \( \frac{1}{0.54} \approx 1.8525 \) under \( r_1 = 1, r_2 = 1 \), and pay 0 otherwise. Yes, it is strictly proper, because the self-predicting property holds (see Theorem 17.4):

\[ P(X_2 = 1|X_1 = 1) > P(X_2 = 1|X_1 = 0) \]

and

\[ P(X_2 = 0|X_1 = 0) > P(X_2 = 0|X_1 = 1) \]

Another way to check this is directly via the expected payoff to each report.

\[ 2.778 \cdot P(X_2 = 0|X_1 = 0) \approx 1.297326 > 1.5625 \cdot P(X_2 = 1|X_1 = 0) \approx 0.8328125 \]
\[ 1.5625 \cdot P(X_2 = 1|X_1 = 1) \approx 1.09382 > 2.778 \cdot P(X_2 = 0|X_1 = 1) \approx 0.8334 \]
2 Prediction Markets

2.1 Review

1. Prediction markets are systems for aggregating the beliefs of many people about one or more future events by allowing agents to place bets on the future outcomes of those events. One example of such a market is the Iowa Electronic Market (one of few operating in the U.S.). The challenge in designing prediction markets is to create incentives such that those with new information or strongly held beliefs will choose to participate in the market.

2. In the prediction market, agents can buy contracts on the outcome of an event. Ideally, if an agent believes the probability of an event to be $p$, then he or she would buy contracts on the event at a price less than $p$ and short the contracts on the event at a price greater than $p$. Types of contracts include winner-take-all and index contracts.

3. A winner-take-all contract pays off if the underlying event occurs, and the contract owner gets nothing if the event does not occur. Then the current market price corresponds to the market’s belief regarding the probability of the event happening. If the contract pays off $10 when the event happens, and the contract is trading at $3$, then the market believes that there is a $30\%$ chance that the event will occur, assuming risk neutrality.

4. Two important prediction market designs are the continuous double auction (CDA) and the automated market maker (AMM). In the CDA, an order book maintains outstanding bids and asks. Orders are matched and carried out whenever there is a pair of a bid and ask such that the bid price is lower than the ask price (the trade is executed at the price of whichever of the bid or ask was submitted first). In the AMM, the prediction market operator acts as a market maker and is always willing to trade. The main advantage of this design is greater liquidity.

5. For AMM, we want a mechanism with the following properties: no round-trip arbitrage, strictly positive prices, normalization, responsiveness, high liquidity, myopic incentives, and bounded loss. The log market scoring rule (LMSR) is a market maker with these properties.

6. Combinatorial prediction markets allow agents to bid on a large number of possible outcomes and the various possible combinations of possible outcomes, such as the presidential primaries. The challenges of designing such a market are maintaining liquidity and creating an expressive bidding language for agents.

7. Prediction markets advantages over classical methods such as asking experts and opinion polls include high predictive accuracy, self-selection, incentive alignment, agents putting money behind their bets, running in real-time, and self-organizing. What are the limitations?

2.2 Exercises

2.2.1 Continuous Double Auctions

We will analyze a market with a Continuous Double Auction mechanism and one contract traded (XYZ). Assume that there are no outstanding orders at the beginning of this question and that prices are set with preference given to earlier orders.
1. A trader, $A_1$, puts in a buy order at $0.50$, and another trader, $A_2$, puts in a sell order at $0.60$. What happens?

2. Another trader, $A_3$, puts in a buy at $0.65$. Now what happens? If a trade occurs, at what price?

3. Another trader, $A_4$, puts in a sell order at $0.40$. Now what happens?

**Answer:**

1. No trade will occur.

2. Since $A_3$ offers to buy at a higher price than the lowest asking price ($0.6$ from $A_2$), these two traders will transact. Typically, the price of the transaction is set according to the earlier order, so the price is $0.6$.

3. $A_1$ buys a contract of XYZ from $A_4$. $A_1$’s bid was on the books first, so $A_1$ buys at their bid, which is $0.50$.

**2.2.2 Automated Market Maker with Market Scoring Rule**

We will analyze a market with an automated market maker using a logarithmic market scoring rule. Suppose that there are two outcomes, $o_0$ and $o_1$, possible, and that the automated market maker uses the cost function

$$C(x_0, x_1) = 2 \ln(e^{x_0} + e^{x_1}).$$

The initial state is $(0, 0)$.

1. A trader, $A_0$, puts in a buy order for 1 contract of $o_0$ occurring. What does he pay?

2. A trader, $A_1$, puts in a buy order for 2 contracts of $o_0$ occurring. What does he pay?

3. Compute the profits or losses of $A_0$ and $A_1$ if outcome $o_0$ occurs and if outcome $o_1$ occurs.

**Answer:**

1. $C(1, 0) - C(0, 0) = 2(\ln(e^{\frac{1}{2}} + e^{0}) - \ln(2e^{0})) \approx 0.56$.

2. $C(3, 0) - C(1, 0) = 2(\ln(e^{\frac{3}{2}} + e^{0}) - \ln(e^{\frac{1}{2}} + e^{0})) \approx 1.45$.

3. If $o_0$ occurs, then $A_0$ makes $1$ and paid $0.56$ for a profit of $.44$. If $o_1$ occurs, then he gets no money and incurs losses of $.56$. $A_1$ makes $.55$ if $o_0$ occurs, and loses $1.45$ if $o_1$ occurs.

**2.2.3 Logarithmic Market Scoring Rule**

Consider the following market making mechanism that is being used to predict the outcome of the 2020 presidential election. There is a first guess probability that a democrat will win with $p=50\%$. Whenever a trader comes along and disagrees with the market’s probability $p_{last}$, he may change the market probability to $p_{new}$. He is paid according to the logarithmic scoring rule on $p_{new}$, but he must pay out the logarithmic scoring rule on $p_{last}$. 
1. What is an agent’s expected payoff if he moves the market from \( p_{\text{last}} \) to his best guess probability \( p_{\text{new}} \) and is scored using the logarithmic scoring rule?

2. The market is currently trading at \( p = .40 \). A new participant who believes a democrat will win with probability \( p = .80 \) comes along. Assuming he is risk neutral and not budget constrained, what value will he move the market to? What is his expected payoff?

3. If you are running this mechanism, what is the most that you can lose?

**Answer:**

1. The agent’s expected payoff is his expected revenue less the costs he incurred, giving us

\[
p_{\text{new}} \cdot \ln(p_{\text{new}}) + (1 - p_{\text{new}}) \cdot \ln(1 - p_{\text{new}}) - p_{\text{new}} \cdot \ln(p_{\text{old}}) - (1 - p_{\text{new}}) \cdot \ln(1 - p_{\text{old}})
\]

2. He will move the market to his true belief, which is \( p_{\text{new}} = .8 \). Plugging into the above, we obtain:

\[
(.8) \cdot \ln(.8) + (.2) \cdot \ln(.2) - (.8) \cdot \ln(.4) - (.2) \cdot \ln(.6) = .334795287
\]

3. Denoting the outcome as \( o \), the total payment made by the market organizer is:

\[
[t(q^n, o) - t(q^{n-1}, o)] + [t(q^n, o) - t(q^{n-1}, o)] + \cdots + [t(q^1, o) - t(q^0, o)]
\]

where \( q^n \) denotes the belief of the market at period \( i \). By the telescoping nature of this sum, this is equivalent to:

\[
= t(q^n, o) - t(q^0, o)
\]

The greatest the first term could be is \( q^n = 1 \), and the initial belief is \( \frac{1}{2} \), so we have

\[
= \ln(1) - \ln(.5) = \ln 2
\]