1 Key Ideas to Understand

1. Two Sided Matching Problems: We have two distinct sets of agents. Each agent wants to either be matched with an agent in the other set or remain unmatched. Agents have a strict preference over all possible candidates from the other side (including the null candidate, or not being matched). An important property of two sided matching is stability. A matching is stable if there is no blocking pair of agents who prefer each other to their assigned match.

2. Deferred Acceptance Algorithm: There does not exist a two-sided matching mechanism that is stable (always outputs a stable matching) and strategy proof for both sides (Roth ‘82). However, we can achieve stability along with strategy-proofness for one side of the market with the deferred acceptance algorithm. Mechanism:
   - Participants on one side of the market (e.g. students) proposes to the participants (e.g. professor) on the other side of the market.
   - Professors decides whether or not to accept the proposed offer depending on her preference. If accepted, and the professor had held an offer, she rejects the student with the previously held offer.
   - If students are rejected, then they move down and propose to their next favorite professor.
   - Algorithm ends when there is a stage where no students are rejected.

Note that
   (a) This algorithm always terminates with a stable matching
   (b) It is only strategy proof for the side proposing. No matching mechanism for two-sided matching is both stable and strategy proof.
   (c) A professor is in the set of achievable professors to a student if there exists some stable matching between the professor and that student. The student proposing DA matches each student their most preferred achievable professor, and matches each professor with their least preferred achievable student.

3. Assignment Problems: Assignment problems are one-sided matching problems, in that there is a single group of agents and a set of items, and agents have preferences on items but items do not have preferences on agents. Two variants of the assignment problem are the House Allocation Problem, in which one item is assigned to each agent, and the Housing Markets Problem, in which each agent initially owns one of the items, and an assignment represents a reallocation of the items. Agents have a strict preference ordering over all items.

4. Serial Dictatorship and Random Serial Dictatorship: Serial Dictatorship fixes an ordering of agents, and let them pick one-by-one what they prefer (out of the options that has not been picked). The mechanism is strategy proof and Pareto optimal. Random Serial Dictatorship mechanism, uniform randomly assigns the ordering. The mechanism is also strategy proof and Pareto optimal for the house allocation problem.
5. **Top Trading Cycles Mechanism**: An assignment in the housing markets problem is said to be in the **core** if there is no set of agents in the assignment that would do better by trading among themselves (i.e. if there is no blocking coalition). The Top Trading Cycle Mechanism selects an assignment that is Pareto-optimal, satisfies participation, strategy proof and in the (unique) core. Mechanism:

- Let agents point to the agent that has their most preferred house that is still in the market.
- Find all cycles, and have each agent receive the house they point to. The agents that are allocated exit the market.
- Repeat until no agents are left.

6. It is important to note that for any of the algorithms, the “proposal” and “acceptance” (DA), “cycle formation” and “pointing” (TTC), and ‘pick one-by-one’ (SD, RSD) are all done implicitly by the mechanism. The agents (students and professors, house traders) only submit a list of their preferences. Then the mechanism, using the algorithm described, spits out the matching allocation.

7. **Kidney Paired Donation**: The Kidney Paired donation problem can be formulated as a problem of finding vertex disjoint cycles on a directed compatibility graph \( G = (V, E) \). The vertices in \( V \) are patient-donor pairs and there is a directed edge \( e = (u, v) \) from a pair \( u \) to a pair \( v \) if the donor in pair \( u \) is compatible with the patient in pair \( v \). The cycles must be vertex disjoint since we assume that each patient is able to donate only one kidney and each patient needs one kidney. The length of each cycle is limited to \( K \) for logistical reasons. When \( K = 2 \), we can formulate this problem as a maximum cardinality matching (which is polynomial time solvable due to Edmond’s algorithm). For \( K = \infty \) this problem can be solved using linear programming. However, for \( 3 \leq K < \infty \) this problem is NP-hard.

## 2 Problems

1. **House Allocation**

   Consider five agents who wish to be assigned a house. The preference of agent \( i \) over all houses is denoted \( \prec_i \) and summarized in the following table (most preferred on top):

   \[
   \begin{array}{cccccc}
   \prec_1 & \prec_2 & \prec_3 & \prec_4 & \prec_5 \\
   2 & 3 & 1 & 5 & 2 \\
   5 & 1 & 3 & 2 & 2 \\
   1 & 4 & 4 & 1 & 4 \\
   4 & 3 & 2 & 1 & 3 \\
   3 & 2 & 4 & 5 & 4 \\
   \end{array}
   \]

   Given a priority order \( \pi(1) = 4, \pi(2) = 3, \pi(3) = 1, \pi(4) = 5, \pi(5) = 2 \) over the agents, use the serial dictatorship mechanism to determine an allocation of the houses to the agents. Argue that this allocation is strategy proof and Pareto optimal.

2. **Housing Market**

   Consider six agents who each own a house. The preference of agent \( i \) over all houses is denoted \( \prec_i \) and summarized in the following table (most preferred on top):

   \[
   \begin{array}{ccccccc}
   \prec_1 & \prec_2 & \prec_3 & \prec_4 & \prec_5 & \prec_6 \\
   2 & 1 & 2 & 4 & 5 & 3 \\
   5 & 6 & 1 & 3 & 3 & 3 \\
   1 & 4 & 4 & 4 & 1 & 4 \\
   6 & 3 & 3 & 1 & 2 & 2 \\
   3 & 2 & 6 & 5 & 5 & 6 \\
   4 & 5 & 5 & 6 & 6 & 1 \\
   \end{array}
   \]
Use the Top Trading Cycles Mechanism to find a Pareto optimal assignment in the core. Then check: i) Agent 5 can’t manipulate the result by misreporting her preference; ii) \{1, 2\}, \{2, 3\}, and \{1, 3, 5\} are not blocking coalitions.

3. Stable Matching I

We have three students \(s_1, s_2, s_3\) and three thesis advisors \(m_1, m_2, m_3\). Each thesis advisor needs exactly one student and each student needs exactly one thesis advisor (no one will remain unmatched). Given the following preferences:

\[
\begin{array}{ccccccc}
\prec_{s_1} & \prec_{s_2} & \prec_{s_3} & \prec_{m_1} & \prec_{m_2} & \prec_{m_3} \\
 m_1 & m_3 & m_3 & s_2 & s_1 & s_1 \\
 m_3 & m_1 & m_2 & s_1 & s_3 & s_2 \\
 m_2 & m_2 & m_1 & s_3 & s_2 & s_3 \\
\end{array}
\]

(a) Find the stable matching where the students propose.

(b) Find the stable matching where the thesis advisors propose.

(c) In the game where the students propose, find a useful misreport from one of the thesis advisors that improves the advisor’s allocation or argue that no such misreport exists.

4. Stable Matching II

Suppose we have the same game as above but with four students, four professors, and the below preferences:

\[
\begin{array}{cccccccc}
\prec_{s_1} & \prec_{s_2} & \prec_{s_3} & \prec_{s_4} & \prec_{m_1} & \prec_{m_2} & \prec_{m_3} & \prec_{m_4} \\
 m_3 & m_4 & m_1 & m_3 & s_1 & s_3 & s_3 & s_3 \\
 m_4 & m_1 & m_3 & m_1 & s_2 & s_2 & s_2 & s_1 \\
 m_1 & m_3 & m_4 & m_4 & s_3 & s_4 & s_4 & s_1 \\
 m_2 & m_2 & m_2 & m_4 & s_4 & s_1 & s_1 & s_2 \\
\end{array}
\]

(a) Find the stable matching where the students propose.

(b) Find the stable matching where the thesis advisors propose.

(c) (Challenge) In the game where the thesis advisors propose, find a useful misreport from one of the students that improves the student’s allocation or argue that no such misreport exists.

5. Paired Kidney Donation

The (incomplete) compatibility graph below has vertices as patient-donor pairs \((P_i, D_i)\) a directed edge from \((P_1, D_1)\) to \((P_2, D_2)\) if \(D_1\) can donate kidney to \(P_2\).

(a) Complete the graph by drawing all the edges from a donor to a compatible patient.

(b) Find a maximum cardinality matching where cycles are limited to 2-cycles.

(c) Now find a maximum cardinality matching where 3-cycles are allowed in addition to 2-cycles.